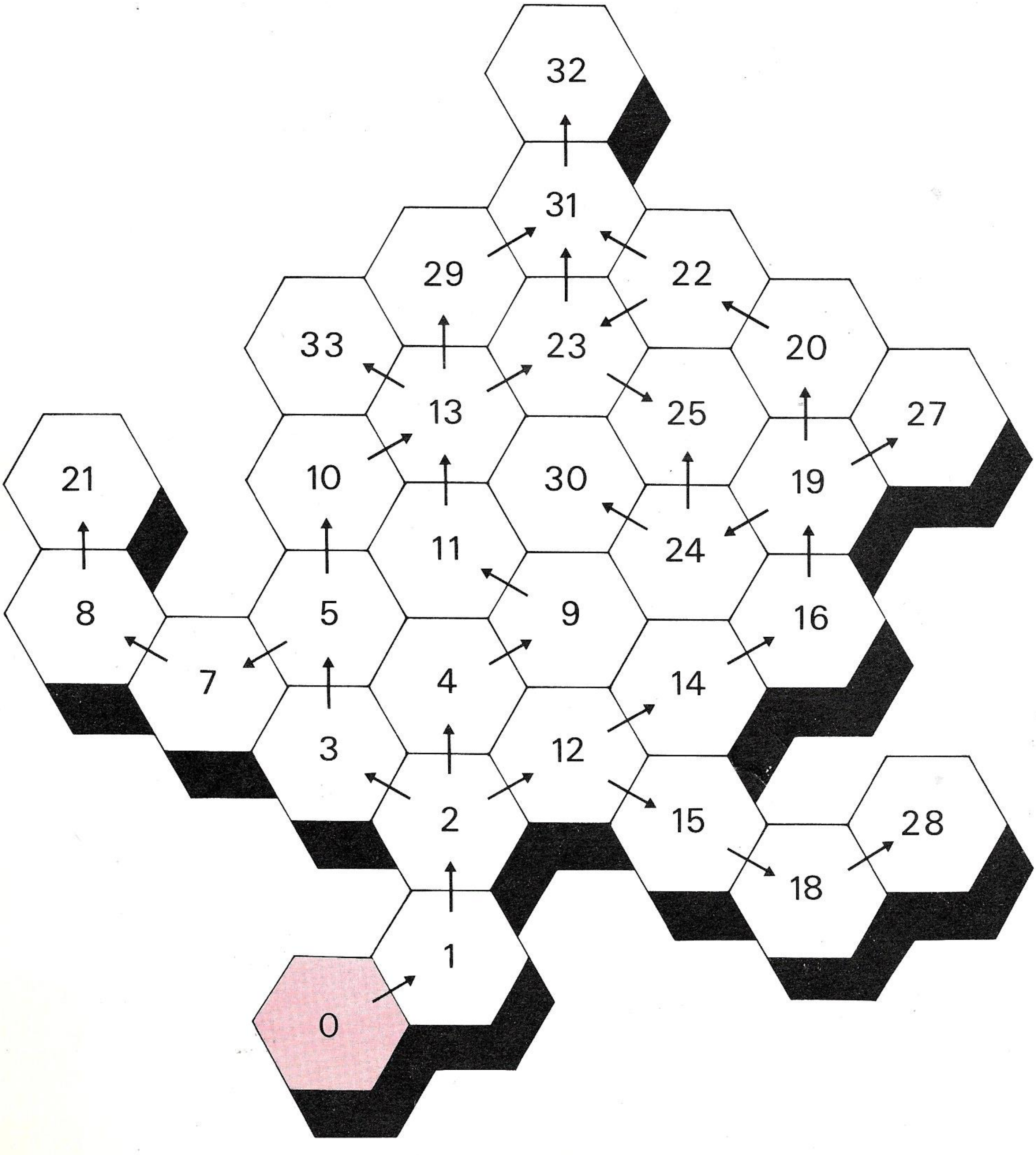
M201 0

THE OPEN UNIVERSITY

Mathematics: A Second Level Course

Linear Mathematics Unit 0

Linear Algebra



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Mathematics: A Second Level Course

Linear Mathematics Unit 0

LINEAR ALGEBRA

Prepared for the Course Team

The Open University Press

The Open University Press Walton Hall Milton Keynes

First published 1978. Reprinted (with corrections) 1979.

Produced in Great Britain by
Technical Filmsetters Europe Ltd.
76 Great Bridgewater Street, Manchester M1 5JY

ISBN 0 335 01125 X

This text forms part of the correspondence element of an Open University Second Level Course. The complete list of units in the course given at the end of this text.

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1.2

Contents

Set Books

D. L. Kreider, R. G. Kuller, D. R. Ostberg and F. W. Perkins, *An Introduction to Linear Analysis* (Addison-Wesley, 1966).
E. D. Nering, *Linear Algebra and Matrix Theory* (John Wiley, 1970).

It is essential to have these books; the course is based on them and will not make sense without them.

Conventions

Before working through this correspondence text make sure you have read *A Guide to the Linear Mathematics Course.* Of the typographical conventions given in the Guide the following are the most important.

The set books are referred to as:

K for *An Introduction to Linear Analysis*
N for *Linear Algebra and Matrix Theory*

Note

This unit is not based on the set books. It has been written especially for the benefit of students who have taken the Mathematics Foundation Course M101 (The Open University Press, 1978).

0.0 INTRODUCTION

This new first unit of the course gives a preview of the fundamental ideas of vector spaces and linear transformations which you will meet again in *Units 1 and 2*. It is of crucial importance for all of your subsequent work on this course that you should acquire a firm understanding of these central concepts: such understanding is best assured by prolonged exposure to the new ideas.

Consider the following problems.

(a) Solve

$$2x + 3y = 8$$
$$x + 4y = 9.$$

(b) Find the point in the plane whose image is (0, 1) when rotated about the origin through the angle $\pi/3$.

(c) Find the general solution of the differential equation

$$q'(t) = I - \lambda q(t).$$

These three seemingly unrelated problems can be solved by methods discussed in the Foundation Course, M101. What they have in common is an element of *linearity* which will become apparent as this course unfolds.

The notion of linearity arises essentially from geometric considerations. In the three-dimensional world, the most common mode of simplifying a problem is to restrict consideration to a *plane* or a *line*. These subsets of space possess a property which we recognize by its lack of curvature; it is this property that we shall seek to characterize. Once we have done that, we shall find that there are many sets of 'objects' in mathematics which can, by a judicious choice of viewpoint, be deemed to possess the property of linearity.

Our aim is to find, for suitable sets, methods by which the 'objects' can be *added* to each other and *scaled* by real numbers. It turns out that these two fundamental notions encapsulate what we mean by linearity.

0.1 LINEARITY IN THE PLANE

0.1.1 The Set R^2

The algebraic view of geometry, which has prevailed in much of the Foundation Course, requires that we abandon geometric intuition in favour of the interpretation of the symbols of algebra. Thus a *point* in the plane is represented by a number pair, say (x, y), which implicitly refers to a predetermined choice of coordinate axes and a unit of measurement. A *curve* is considered as a subset of the plane consisting of all the points whose coordinates satisfy some equation such as

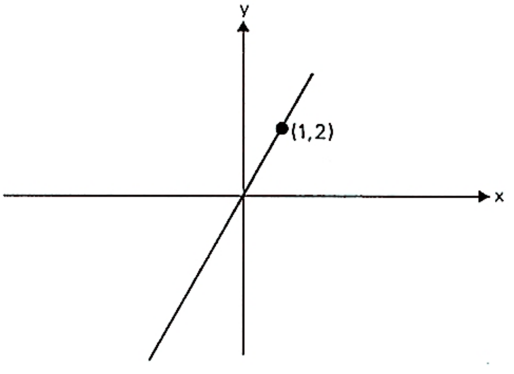
$$\{(x, y): x^2 + y^2 = 1\}$$

or

$$\{(x, y): 4x + 3y = 2\}.$$

The first of these is a circle and the second a straight line, but how can we tell without actually sketching the curves?

Let us simplify matters by considering only lines through the origin.



The line through the origin which also contains $(1, 2)$ consists of all points whose coordinates are of the form $(\lambda, 2\lambda)$ for some $\lambda \in R$.

In this course we do not use the open typeface for the standard sets. Thus the set of reals is denoted by R rather than $\mathbb{R}$.

The equation of this line is easily seen to be

$$\{(x, y): y - 2x = 0\}.$$

From the algebraic viewpoint, the distinguishing characteristic of straight line is that its equation can involve the following two operations:

multiplication of a coordinate *by a real number* (or *scaling*),

addition.

This infallible rule tells us immediately that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

cannot be a straight line, on account of the x^2 and y^2, even if we have never heard of an ellipse.

Our aim now is to extend the arithmetic operations of scaling and addition to other sets. We might as well start with the whole plane, or rather the set R^2 which is defined as the set of all real-number pairs.

$$R^2 = \{(x, y): x \in R \quad \text{and} \quad y \in R\}.$$

How shall we define addition and scaling on R^2? Well, we have already had cause to consider the line passing through the origin and (x, y), which consists of points whose coordinates are $(\lambda x, \lambda y)$ and we shall define scalar multiplication of a pair by λ according to the rule

$$\lambda(x, y) = (\lambda x, \lambda y)$$

Thus the real number λ scales the pair (x, y) in R^2 by multiplying each coordinate separately. Likewise, if we have two pairs (x_1, y_1) and (x_2, y_2) we define addition of pairs according to the rule

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

Compare this with addition of complex numbers, defined in M101 Block VI *Unit 1*, Section 1.2.

Note that although we use the same plus symbol on both sides of this defining equation, its meaning on the right-hand side is unambiguously the familiar addition of reals which we know so well, whereas on the left-hand side the $+$ serves as a newly defined operator on members of the set R^2.

Exercises

In this course the solutions will be printed immediately following the exercises. It would be advisable to keep a size A5 card handy, to cover the solution while you are working an exercise.

1. Evaluate the following:

   (i) $(1, 0) + (3, -\frac{1}{2})$
   (ii) $(5, -1) + (-1, 1\frac{1}{3})$
   (iii) $(a, b) + (a, 0)$
   (iv) $(c, d) + (-c, -d)$

2. Evaluate the following:

   (i) $3(1, 7)$
   (ii) $4\frac{1}{2}(0, 0)$
   (iii) $1(a, b)$

3. Evaluate the following:
   (i) $-1(3\frac{1}{2}, 2) + (-3, 2)$
   (ii) $7(1, -1) + 0(2, \pi)$
   (iii) $4(-1, 4) + 8(\frac{1}{2}, -2)$
   (iv) $\lambda(1, 0) + \mu(0, 1)$

Solutions

1. (i) $(1, 0) + (3, -\frac{1}{2}) = (1 + 3, 0 - \frac{1}{2}) = (4, -\frac{1}{2})$.
   (ii) $(4, \frac{1}{3})$ (iii) $(2a, b)$ (iv) $(0, 0)$.
2. (i) $(3, 21)$ (ii) $(0, 0)$ (iii) (a, b).
3. (i) $(-6\frac{1}{2}, 0)$ (ii) $(7, -7)$ (iii) $(0, 0)$ (iv) (λ, μ).

0.1.2 Linear Transformations on the Plane

The operations of addition and scalar multiplication endow R^2 with some structure. We could now proceed with the investigation of this structure (as indeed we shall do later on), but first it will not come amiss if we satisfy ourselves that we are going to achieve something by imposing a mathematical structure on R^2, over and above the ability to play little arithmetical games with number pairs.

Suppose T denotes the transformation $R^2 \longrightarrow R^2$ which rotates each point of R^2 through an angle θ anticlockwise round the origin. It may be shown that

$$T: (u, v) \longmapsto (u \cos\theta - v \sin\theta, u \sin\theta + v \cos\theta).$$

M101 Block IV *Unit 3*, Section 3.4.

Now, geometrically, it is clear that a rotation of R^2 maps straight lines to straight lines, in other words, it preserves the *linear structure* of the plane. Algebraically, the idea of a function preserving structure is expressed by the image of a combination being the corresponding combination of the images.

M101 Block VI *Unit 4*.

Exercises

1. What are the images of $(1, 0)$ and $(0, 1)$ under T?
2. What is the image of $(3, -2)$ under T?

3. Express $(3, -2)$ as a combination of $(1, 0)$ and $(0, 1)$ using the operations of addition and scalar multiplication.

Solutions

1. $T(1,0) = (\cos\theta, \sin\theta)$; $T(0,1) = (-\sin\theta, \cos\theta)$.
2. $T(3,-2) = (3\cos\theta + 2\sin\theta, 3\sin\theta - 2\cos\theta)$.
3. $(3,-2) = 3(1,0) - 2(0,1)$.

We have just seen that $(3, -2)$ can be expressed as the combination

$$3(1,0) - 2(0,1).$$

Because the operations of addition and scalar multiplication used to obtain this combination are linear operations, we say that $(3, -2)$ is hereby expressed as a linear combination of $(1, 0)$ and $(0, 1)$. The image of this linear combination is given by

$$\begin{aligned} T(3,-2) &= (3\cos\theta + 2\sin\theta, 3\sin\theta - 2\cos\theta) \\ &= 3(\cos\theta, \sin\theta) - 2(-\sin\theta, \cos\theta). \\ &= 3T(1,0) - 2T(0,1) \end{aligned}$$

In other words the image of $(3, -2)$ is the same linear combination of the images $(\cos\theta, \sin\theta)$ and $(-\sin\theta, \cos\theta)$ as $(3, -2)$ is of $(1, 0)$ and $(0, 1)$. In fact, the specific numbers $3, -2$ were not intrinsic to our argument and, in a like way, we could have shown that for any number pair $(\lambda, \mu) \in R^2$,

$$(\lambda, \mu) = \lambda(1, 0) + \mu(0, 1)$$

and

$$T(\lambda, \mu) = \lambda T(1,0) + \mu T(0,1).$$

This is essentially the map-reference property of M101 Block IV *Unit 4*, Section 4.3.

So the rotation T preserves the linear structure of R^2. This result says, effectively, that provided we know which elements of R^2 are the images of $(1, 0)$ and $(0, 1)$ we can determine the image of an arbitrary element (λ, μ) as the appropriate linear combination of those images.

We can develop this further more easily if we introduce the notation

$$\boldsymbol{i} = (1, 0) \qquad \boldsymbol{j} = (0, 1)$$

The equals sign in these formulas mean, e.g., that $\boldsymbol{i}$ is *another name* for $(1, 0)$.

to avoid having to write these number pairs in full each time. A typical element of R^2 is

$$(\lambda, \mu) = \lambda\boldsymbol{i} + \mu\boldsymbol{j}.$$

The images of $\boldsymbol{i}$ and $\boldsymbol{j}$ under T are

$$T(\boldsymbol{i}) = (\cos\theta, \sin\theta) = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$$

and

$$T(\boldsymbol{j}) = (-\sin\theta, \cos\theta) = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}.$$

The linear nature of rotation manifests itself in the formula

$$T(\lambda\boldsymbol{i} + \mu\boldsymbol{j}) = \lambda T(\boldsymbol{i}) + \mu T(\boldsymbol{j}).$$

In the Foundation Course we observed this very result by a somewhat different means, using matrices. In fact, the matrix approach is intimately connected with this formula. We write a *column matrix* instead of using $\boldsymbol{i}$ and $\boldsymbol{j}$, as follows:

$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}$ represents (λ, μ)

$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ represents $T(\boldsymbol{i})$

$\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ represents $T(\boldsymbol{j})$.

The rule for matrix multiplication gives us

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}=\begin{bmatrix}\lambda\cos\theta\\ \lambda\sin\theta\end{bmatrix}+\begin{bmatrix}-\mu\sin\theta\\ \mu\cos\theta\end{bmatrix}$$

where the given matrix represents our rotation T. It is no accident that the columns of the 2×2 matrix representing T are identical with the 2×1 column matrices representing $T(i)$ and $T(j)$, as you can see in the next exercise.

Exercise

Multiply out

(i) $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$ (ii) $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}$

Solution

(i) $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ (ii) $\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$

0.1.3 Summary of Section 0.1

In this section we have defined the terms

addition (of pairs) (page 7)

scalar multiplication (of pairs). (page 6)

We introduced the notation

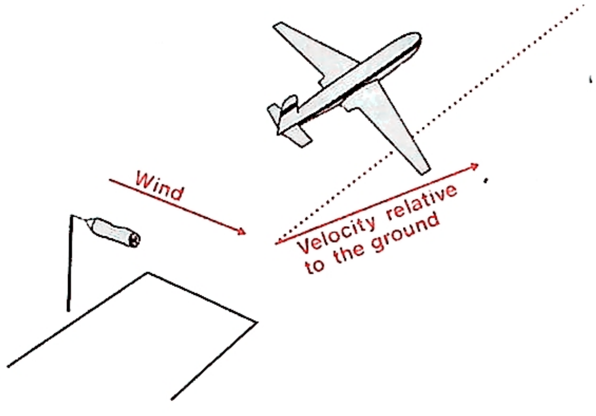
R^2. (page 6)

You are not expected to remember every detail of this section; we have attempted here only to take an example from within your experience to set the stage for the study ahead of us. This study will involve us in constructing the linear operations of addition and scalar multiplication on certain suitable sets and considering the properties of transformations between such sets which respect this linearity.

0.2 GEOMETRIC VECTORS

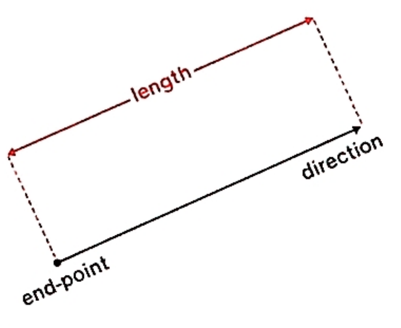
0.2.0 Introduction

The speed and direction of an aeroplane over the ground depend not only upon the thrust of its engines but also upon the strength and direction of the wind. A cross-wind will blow the aircraft off course unless the pilot heads slightly into the wind to compensate. This is a basic problem of navigation.



To construct a mathematical model of such a situation, we represent the wind's effect by a *directed line segment* or *arrow* pointing in the direction of the wind and of length proportional to the wind speed. The velocity of the aircraft through the air is similarly represented by another arrow.

The navigational problem of determining the aircraft's course over the ground then becomes a mathematical problem: *how to combine the arrows.*



Similarly the effect of a force may be modelled by an arrow, whose length and direction correspond to that of the force. The resultant of two forces may be determined from the *parallelogram of forces*, which is a geometric method of *combining the two corresponding arrows.*

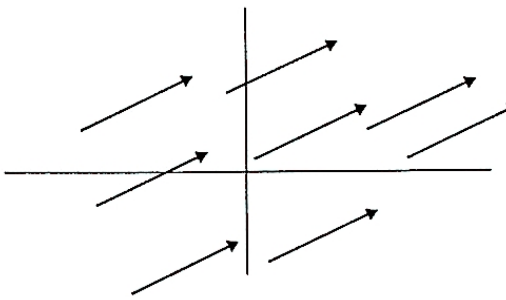
M101 Block V *Unit 4*, Section 4.3.

0.2.1 Translations

Translations were discussed in M101, Block I *Unit 3* and Block IV *Unit 3*.

We shall, by looking at *translations of the plane*, obtain a convenient means of discussing the mathematical properties of physical quantities such as velocity and force which can be modelled geometrically by arrows.

A translation of the plane moves each point through a fixed distance in a fixed direction. To picture a translation, we might indicate its effect on several points of the plane.



Each arrow here has the same length and the same direction; and each indicates the effect of the same translation on the point at which its tail lies, moving it to the position occupied by its head. (If we regard each arrow as indicating the motion of a particle of air over a fixed time-interval, then what we can have here is a constant wind pattern.)

A translation, then, is represented by a set of arrows all sharing the same length and the same direction.

Another way of looking at this is to define a relation between arrows, such that two arrows are related if and only if they have the same length and the same direction.

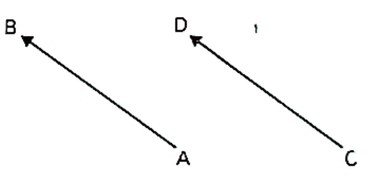
This is easily seen to be an equivalence relation on the set of all arrows in the plane, but we shall not stop to prove it here.

A discussion of equivalence relations can be found in M101 Block IV *Unit 2*, Section 2.3.

Under this relation, each equivalence class is called a geometric vector. Such a geometric vector is a set of arrows: the set of all arrows sharing one particular length and one particular direction. It therefore corresponds exactly to a particular translation of the plane. Strictly speaking, a translation is *represented* by a geometric vector—but the two concepts share the same mathematical properties and we shall regard them as interchangeable.

Clearly a geometric vector is uniquely determined by specifying any one of the arrows belonging to it.

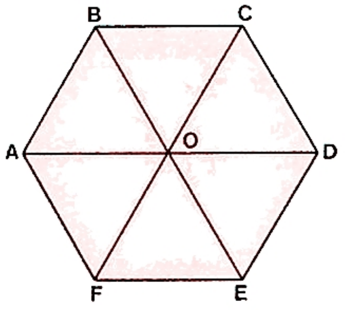
We shall write $\underrightarrow{AB}$ to denote the geometric vector containing the arrow from A to B—but note that $\underrightarrow{AB}$ is not fixed in the position AB. This is just another way of saying that $\underrightarrow{AB}$ represents the translation of the plane which transforms A to B.



The two arrows shown in the diagram above belong to the same geometric vector, so we may write

$$\underrightarrow{AB} = \underrightarrow{CD}.$$

Exercise



The above figure is regular hexagon. We have omitted the arrow heads as these are implied in the following statements. In each case indicate if the statement is true or false:

(i) $\underrightarrow{AB} = \underrightarrow{ED}$ TRUE/FALSE

(ii) $\underrightarrow{FO} = \underrightarrow{ED}$ TRUE/FALSE

(iii) $\underrightarrow{AO} = \underrightarrow{EF}$ TRUE/FALSE

(iv) $\underrightarrow{BC} = \underrightarrow{AD}$ TRUE/FALSE

Solution

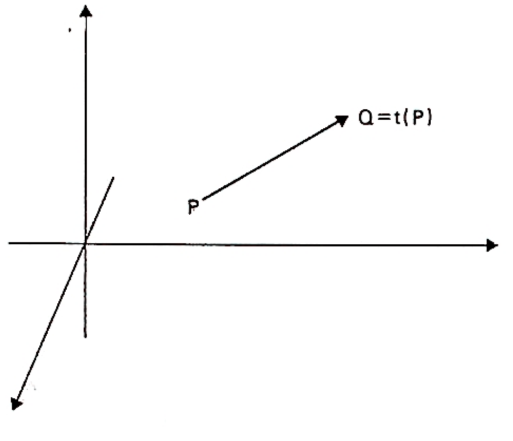
(i) TRUE. The same translation which takes A to B takes E to D, so both arrows represent (belong to) the same geometric vector.

(ii) TRUE. Both the arrows FO and ED belong to the same geometric vector.

(iii) FALSE. The arrows AO and EF (representatives of $\underrightarrow{AO}$ and $\underrightarrow{EF}$ respectively) are in opposite directions.

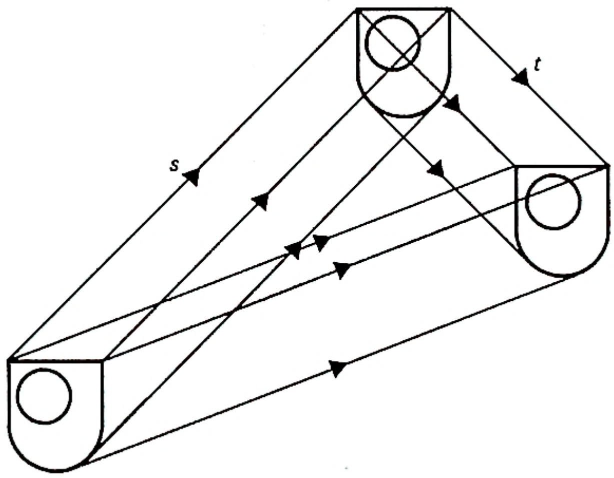
(iv) FALSE. The arrows BC and AD have different lengths.

It is convenient to work with examples in the plane, but we could equally well have geometric vectors in three-dimensional space. If t denotes a translation of three-dimensional space, then to each point P there is a unique point Q such that t is represented by $\underrightarrow{PQ}$.



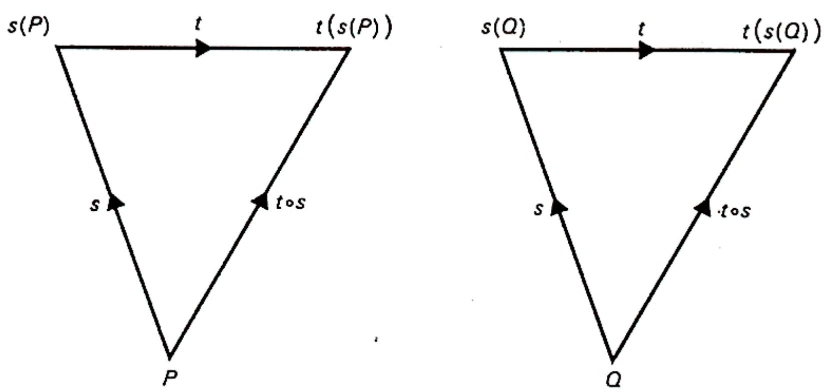
0.2.2 Addition of Geometric Vectors

We compose two translations by first performing one translation and then performing the other. The composition of translations induces a binary operation on the set of geometric vectors, and we call this operation addition. When two translations, s and t say, are composed, the result is another translation.



For convenience we shall sometimes put the arrowhead in the middle of an arrow.

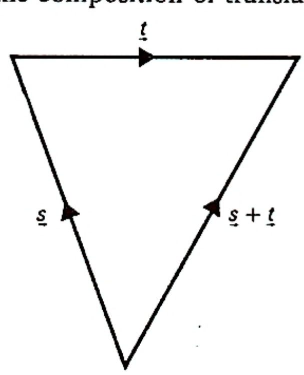
The best way to see this is to consider the effect of the composition $t \circ s$ of the two translations s and t on a general point P in the plane. We can represent s by an arrow with its tail at P, and t by an arrow with its tail at $f(P)$. The effect on P of s, then t—that is, $t \circ s$—is then represented by the arrow from P to $t(s(P))$. Likewise, if we consider the effect of $t \circ s$ on any other point Q then we obtain a triangle of arrows whose sides are parallel and of equal length to the corresponding sides in the triangle obtained at P. In particular, any point Q is carried by $t \circ s$ through the same distance and in the same direction as P, that is, $t \circ s$ is a translation.



If we now call the associated geometric vectors $\underset{\to}{s}$ and $\underset{\to}{t}$, then the triangle of arrows defines a resultant geometric vector $\underset{\to}{u}$, say. We write

$$\underset{\to}{s} + \underset{\to}{t} = \underset{\to}{u}$$

and call u the *resultant* of $\underset{\to}{s}$ and $\underset{\to}{t}$. The rule for addition of geometric vectors is uniquely defined by the composition of translations.



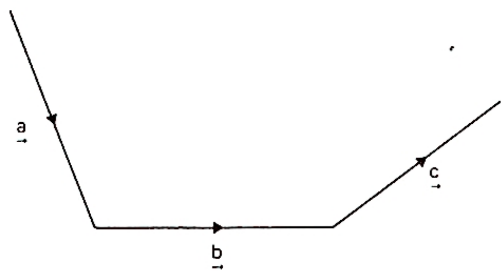
The choice of the symbol + to denote the binary operation which yields the resultant of two geometric vectors is not arbitrary: the properties of addition of real numbers are also properties of the addition of geometric vectors. Two such properties are considered in the exercises which follow.

Exercises

1. Draw a diagram to illustrate the geometric vectors $\underline{a} + \underline{b}$ and $\underline{b} + \underline{a}$. Are the two geometric vectors equal? Is addition of geometric vectors commutative?

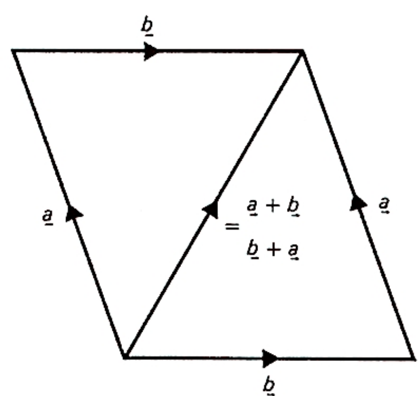
2. Use the following diagram to illustrate the associative property of addition of geometric vectors:

$$(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$$



Solutions

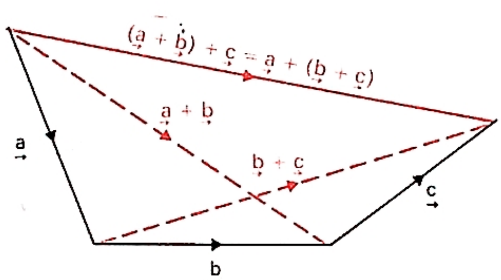
1.



It is true that $\underline{a} + \underline{b} = \underline{b} + \underline{a}$, and therefore addition is commutative. This diagram shows an alternative way of describing the addition of two geometric vectors: if $\underline{a}$ and $\underline{b}$ are represented by two arrows with their tails at the same point then, on completion of the parallelogram, the sum of the two geometric vectors is represented by the diagonal arrow.

Compare with the parallelogram of forces in M101 Block V *Unit 4*.

2.



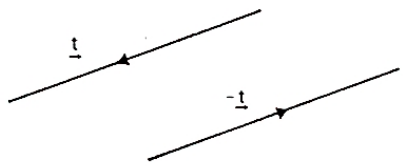
To continue the analogy we would like to define a geometric vector having the property of the number zero for addition in R. A translation through zero distance in any direction carries each point of the plane to itself, that is, it is the identity transformation. Composition of any other translation t with the identity transformation has the same effect as t itself; the

corresponding result for geometric vectors is

$$\underset{\to}{t} + \underset{\to}{0} = \underset{\to}{t} = \underset{\to}{0} + \underset{\to}{t}$$

where $\underset{\to}{0}$ is the geometric vector whose arrows have zero length (and arbitrary direction.) We may call $\underset{\to}{0}$ the zero or *null* geometric vector.

Associated with any translation t we can perform the *inverse* translation which carries each point $t(P)$ back to P. The inverse of t is a translation in the direction opposite to that of t and through the same distance as t. If $\underset{\to}{t}$ is the geometric vector corresponding to the translation t, we write $-\underset{\to}{t}$ for the geometric vector corresponding to the inverse translation.



It should be clear that

$$\underset{\to}{t} + (-\underset{\to}{t}) = \underset{\to}{0}.$$

In line with the spirit of M101 Block VI we now present a summary of the additive properties of geometric vectors.

A1 $\underset{\to}{a} + \underset{\to}{b}$ is a geometric vector. (+ is CLOSED)

A2 $\underset{\to}{a} + (\underset{\to}{b} + \underset{\to}{c}) = (\underset{\to}{a} + \underset{\to}{b}) + \underset{\to}{c}$ (+ is ASSOCIATIVE)

A3 There is a geometric vector, denoted $\underset{\to}{0}$, such that for all $\underset{\to}{a}$

$$\underset{\to}{a} + \underset{\to}{0} = \underset{\to}{a}. \qquad \text{(IDENTITY ELEMENT for +)}$$

A4 For each geometric vector $\underset{\to}{a}$ there exists a (unique) geometric vector $-\underset{\to}{a}$, such that

$$\underset{\to}{a} + (-\underset{\to}{a}) = \underset{\to}{0}. \qquad \text{(existence of INVERSES)}$$

A5 $\underset{\to}{a} + \underset{\to}{b} = \underset{\to}{b} + \underset{\to}{a}$ (+ is COMMUTATIVE)

We now recognize that addition of geometric vectors has the structure of a *commutative group*.

M101 Block VI *Unit 2*.

It is sometimes convenient to have a notation for subtraction of vectors: we define

$$\underset{\to}{a} - \underset{\to}{b} = \underset{\to}{a} + (-\underset{\to}{b}).$$

Here the equals sign means that $\underset{\to}{a} - \underset{\to}{b}$ is another way of writing $\underset{\to}{a} + (-\underset{\to}{b})$

The example which follows illustrates a method of manipulating equations involving geometric vectors.

Example

From the definitions, prove that

$$(\underset{\to}{a} - \underset{\to}{b} = \underset{\to}{c}) \quad \Rightarrow \quad (\underset{\to}{a} = \underset{\to}{c} + \underset{\to}{b}).$$

The symbol $\Rightarrow$ is read "implies" and means that if the left-hand statement is true then the right-hand statement must also be true.

Solution

$$\begin{aligned}
(\underset{\to}{a} - \underset{\to}{b} = \underset{\to}{c}) &\Rightarrow \underset{\to}{a} + (-\underset{\to}{b}) = \underset{\to}{c} && \text{(definition of subtraction)}\\
&\Rightarrow (\underset{\to}{a} + (-\underset{\to}{b})) + \underset{\to}{b} = \underset{\to}{c} + \underset{\to}{b} &&\\
&\Rightarrow \underset{\to}{a} + ((-\underset{\to}{b}) + \underset{\to}{b}) = \underset{\to}{c} + \underset{\to}{b} && \text{(associativity of +)}\\
&\Rightarrow \underset{\to}{a} + \underset{\to}{0} = \underset{\to}{c} + \underset{\to}{b} && \text{(definition of } -\underset{\to}{b})\\
&\Rightarrow \underset{\to}{a} = \underset{\to}{c} + \underset{\to}{b}. && \text{(property of } \underset{\to}{0})
\end{aligned}$$

In particular, if two geometric vectors $\underline{a}$ and $\underline{b}$ satisfy

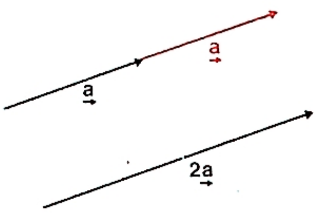
$$\underline{a} - \underline{b} = \underline{0},$$

then $\underline{a}$ and $\underline{b}$ are equal (that is, they represent the same translation).

0.2.3 Scalar Multiples of Geometric Vectors

Any geometric vector has an associated length and direction. Let us consider for a moment the geometric vectors sharing one particular direction. If we compose a translation with itself we obtain another translation in the same direction. We may ask:

what is the length of $\underline{a} + \underline{a}$ in terms of $\underline{a}$?



The translation corresponding to $\underline{a} + \underline{a}$ carries points through twice the length of $\underline{a}$ and in the same direction. This suggests that we write $2\underline{a}$ to mean $\underline{a} + \underline{a}$, the geometric vector having the direction as $\underline{a}$ but twice the length. We can generalize this idea to define $\lambda\underline{a}$, for any real $\lambda > 0$, to mean the geometric vector corresponding to a translation through a distance λ times that of $\underline{a}$ and in the same direction. When $\lambda = 0$ we obtain the geometric vector corresponding to zero distance, that is,

$$0\underline{a} = \underline{0}.$$

When $\lambda < 0$ we may define $\lambda\underline{a}$ to be the translation through the (positive) distance $-\lambda$ in the direction *opposite* to that of $\underline{a}$. This ties in well with the definition of $-\underline{a}$; indeed

$$(-1)\underline{a} = -\underline{a}.$$

This completes the definition of our second operation on geometric vectors, which we call scalar multiplication. We say that $\underline{a}$ has been scaled by λ or *multiplied by the scalar* λ. Unlike addition of geometric vectors, scalar multiplication is defined not between two geometric vectors but between a scalar (or real number) λ and a geometric vector $\underline{a}$. Throughout this course we shall be considering sets of "vectors" which can be "multiplied" or "scaled" by real numbers; in such a context we shall refer to the real numbers as scalars.

Note that we have adopted the usual algebraic convention of representing multiplication by juxtaposing the scalar and the vector, avoiding the need for a multiplication symbol. Scalar multiplication of geometric vectors has the following properties.

B1 $\lambda\underline{a}$ is a geometric vector for each $\lambda \in R$.

B2 $\lambda(\mu\underline{a}) = (\lambda\mu)\underline{a}$

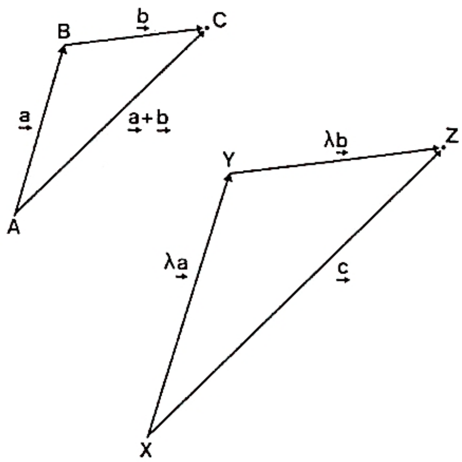
B2 $\lambda(\underline{a} + \underline{b}) = \lambda\underline{a} + \lambda\underline{b}$

B4 $(\lambda + \mu)\underline{a} = \lambda\underline{a} + \mu\underline{a}$

B5 $1\underline{a} = \underline{a}$

These properties follow from the definition of scalar multiplication; we shall demonstrate the proof for *B3*.

If $\lambda = 0$, there is not much to demonstrate. Suppose $\lambda > 0$; then we have the following diagram, where $\vec{c} = \lambda\vec{a} + \lambda\vec{b}$. We want to show that $\vec{c} = \lambda(\vec{a} + \vec{b})$.



Since $\dfrac{XY}{AB} = \dfrac{YZ}{BC} = \lambda$, and angle $ABC =$ angle XYZ, the two triangles are similar. Therefore

$$\frac{XZ}{AC} = \lambda$$

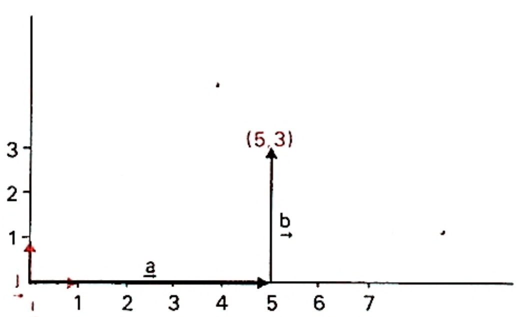
Further, XZ is parallel to AC, and therefore $\vec{XZ} = \lambda\vec{AC}$, i.e.

$$\vec{c} = \lambda(\vec{a} + \vec{b}).$$

If $\lambda < 0$, we have a similar argument.

0.2.4 Linear Dependence and Independence

Consider an ordinary (rectangular Cartesian) coordinate system in the plane.



The translation taking the origin $(0,0)$ into the point $(5,3)$, say, can be accomplished by composing two translations, one taking the origin to the point $(5,0)$, the other taking the origin to the point $(0,3)$. These are translations parallel to the x-axis and y-axis respectively.

Let the corresponding geometric vectors be $\vec{a}$ and $\vec{b}$ respectively, and let $\vec{i}$ and $\vec{j}$ be "unit" geometric vectors parallel to the axes as shown in the diagram.

Since $\vec{a}$ and $\vec{i}$ share the same direction, they differ only in length. Since the distance from the origin to $(5,0)$ is five times the distance to $(1,0)$ the

corresponding translations are related in the way which we used to define scalar multiplication, and so $\underset{\rightarrow}{a} = 5\underset{\rightarrow}{i}$. Similarly we can see that $\underset{\rightarrow}{b} = 3\underset{\rightarrow}{j}$. Therefore

$$\underset{\rightarrow}{a} + \underset{\rightarrow}{b} = 5\underset{\rightarrow}{i} + 3\underset{\rightarrow}{j},$$

and this corresponds to the translation which takes the origin to the point (5, 3).

For the present, it is sufficient that you should have an intuitive grasp of all this. You can probably see now that the translation which takes the origin to the point (x, y) corresponds to the geometric vector $x\underset{\rightarrow}{i} + y\underset{\rightarrow}{j}$. There is thus a unique correspondence between geometric vectors and points in a Cartesian coordinate system.

Moreover, any geometric vector in the plane can be expressed in the form $\lambda\underset{\rightarrow}{i} + \mu\underset{\rightarrow}{j}$, where (λ, μ) is the image of the origin under the corresponding translation. An expression such as $\lambda\underset{\rightarrow}{i} + \mu\underset{\rightarrow}{j}$ is called a linear combination of $\underset{\rightarrow}{i}$ and $\underset{\rightarrow}{j}$.

In general, given geometric vectors $\underset{\rightarrow}{a}_1, \ldots, \underset{\rightarrow}{a}_m$, we can consider linear combinations of the form

$$\lambda_1\underset{\rightarrow}{a}_1 + \cdots + \lambda_m\underset{\rightarrow}{a}_m$$

for scalars $\lambda_1, \ldots, \lambda_m$.

If we are given a set $\{\underset{\rightarrow}{a}_1, \ldots, \underset{\rightarrow}{a}_m\}$ with the property that *every* geometric vector in the plane can be expressed as such a linear combination, then the set $\{\underset{\rightarrow}{a}_1, \ldots, \underset{\rightarrow}{a}_m\}$ spans the set of all geometric vectors in the plane.

Thus, $\{\underset{\rightarrow}{i}, \underset{\rightarrow}{j}\}$ spans the set of (planar) geometric vectors, since each geometric vector can be expressed as a linear combination

$$\lambda\underset{\rightarrow}{i} + \mu\underset{\rightarrow}{j}.$$

In the examples and exercises which follow, we shall consider the geometric vectors given by:

$$\underset{\rightarrow}{a} = \underset{\rightarrow}{i} + 2\underset{\rightarrow}{j}$$

$$\underset{\rightarrow}{b} = \underset{\rightarrow}{i} + \underset{\rightarrow}{j}$$

$$\underset{\rightarrow}{c} = 2\underset{\rightarrow}{i}$$

$$\underset{\rightarrow}{d} = 2\underset{\rightarrow}{i} + 2\underset{\rightarrow}{j}.$$

Example 1

Does the set $\{\underset{\rightarrow}{a}, \underset{\rightarrow}{b}, \underset{\rightarrow}{c}\}$ span the set of planar geometric vectors?

Solution

If $\underset{\rightarrow}{v}$ is a geometric vector in the plane we know that we can find scalars $\lambda, \mu \in R$ such that

$$\underset{\rightarrow}{v} = \lambda\underset{\rightarrow}{i} + \mu\underset{\rightarrow}{j}.$$

We note that

$$\underset{\rightarrow}{a} + \underset{\rightarrow}{b} - \underset{\rightarrow}{c} = 3\underset{\rightarrow}{j}$$

and

$$\underset{\rightarrow}{c} = 2\underset{\rightarrow}{i}$$

so that

$$\underset{\rightarrow}{i} = \tfrac{1}{2}\underset{\rightarrow}{c}$$

and

$$\underset{\rightarrow}{j} = \tfrac{1}{3}(\underset{\rightarrow}{a} + \underset{\rightarrow}{b} - \underset{\rightarrow}{c}).$$

Hence

$$\begin{aligned}\underline{v} &= \lambda\,\underline{i} + \mu\underline{j}\\ &= \frac{\lambda}{2}\underline{c} + \frac{\mu}{3}(\underline{a} + \underline{b} - \underline{c})\\ &= \frac{\mu}{3}\underline{a} + \frac{\mu}{3}\underline{b} + \left(\frac{\lambda}{2} - \frac{\mu}{3}\right)\underline{c};\end{aligned}$$

that is, any $\underline{v}$ can be expressed as a linear combination of $\underline{a}$, $\underline{b}$ and $\underline{c}$. In other words, $\underline{a}$, $\underline{b}$ and $\underline{c}$ spans the set of geometric vectors.

Exercises

1. Express $3\underline{i} + 2\underline{j}$ as a linear combination of $\underline{a}$, $\underline{b}$ and $\underline{c}$.
2. Show that $\{\underline{b}, \underline{i}\}$ spans the set of geometric vectors.
3. Show that $\{\underline{b}, \underline{d}\}$ does not span the set of all geometric vectors. Which subset of geometric vectors is spanned by $\{\underline{b}, \underline{d}\}$?

Solutions

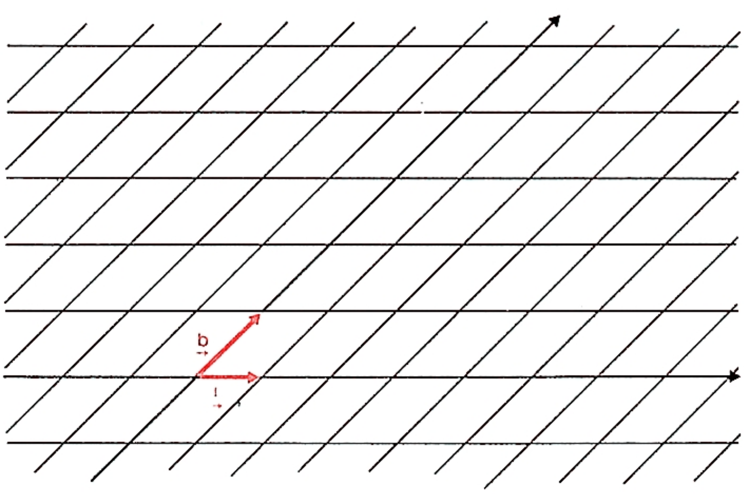
1. Using the formula obtained in the example, we obtain

$$3\underline{i} + 2\underline{j} = \tfrac{2}{3}\underline{a} + \tfrac{2}{3}\underline{b} + \tfrac{5}{6}\underline{c}.$$

2. Since $\underline{b} = \underline{i} + \underline{j}$, we have

$$\lambda\,\underline{i} + \mu\,\underline{j} = \mu\,\underline{b} + (\lambda - \mu)\,\underline{i}.$$

This corresponds to an oblique coordinate system, as shown in the figure.



3. It is sufficient to find ONE geometric vector which *cannot* be expressed as a linear combination $\alpha\underline{b} + \beta\underline{d}$. Now

$$\alpha\underline{b} + \beta\underline{d} = (\alpha + 2\beta)(\underline{i} + \underline{j}),$$

So it is clear that no choice of $\alpha, \beta \in R$ will give

$$\alpha\,\underline{b} + \beta\,\underline{d} = \underline{i},$$

for example.

In fact,

$$\begin{aligned}\alpha\,\underline{b} + \beta\,\underline{d} &= \alpha\,\underline{b} + 2\beta\,\underline{b}\\ &= (\alpha + 2\beta)\,\underline{b}\end{aligned}$$

(since $\underline{d} = 2\underline{b}$).

So $\{\underset{\to}{b}, \underset{\to}{d}\}$ spans only the subset of geometric vectors parallel to $\underset{\to}{b}$ (i.e. the scalar multiples of $\underset{\to}{b}$).

The equation $\underset{\to}{d} = 2\underset{\to}{b}$ can be rearranged to

$$2\underset{\to}{b} - \underset{\to}{d} = \underset{\to}{0},$$

which expresses $\underset{\to}{0}$ as a (nontrivial) linear combination of $\underset{\to}{b}$ and $\underset{\to}{d}$.

Whenever there exist scalars $\lambda_1, \ldots, \lambda_m$, not all zero, such that

$$\lambda_1 \underset{\to}{a}_1 + \cdots + \lambda_m \underset{\to}{a}_m = \underset{\to}{0}$$

we say that the set of geometric vectors $\{\underset{\to}{a}_1, \ldots, \underset{\to}{a}_m\}$ is linearly dependent.

Otherwise, if $\lambda_1 a_1 + \cdots + \lambda_m a_m = \underset{\to}{0}$ implies that $\lambda_1 = \cdots = \lambda_m = 0$, the set $\{a_1, \ldots, a_m\}$ is said to be linearly independent. The simplest example of a linearly independent set is $\{\underset{\to}{a}\}$ where $\underset{\to}{a}$ is any geometric vector other than $\underset{\to}{0}$. Another simple example is $\{\underset{\to}{i}, \underset{\to}{j}\}$, for the only linear combination of $\underset{\to}{i}$ and $\underset{\to}{j}$ which can yield $\underset{\to}{0}$ corresponding to the zero translation) is $0\underset{\to}{i} + 0\underset{\to}{j}$.

Example 2

$\{\underset{\to}{b}, \underset{\to}{d}\}$ is a linearly dependent set, since $\underset{\to}{d} - 2\underset{\to}{b} = \underset{\to}{0}$.

Example 3

$\{\underset{\to}{b}, \underset{\to}{i}\}$ is a linearly independent set, for if $\alpha\underset{\to}{b} + \beta\underset{\to}{i} = \underset{\to}{0}$ then

$$(\alpha + \beta)\underset{\to}{i} + \alpha\underset{\to}{j} = \underset{\to}{0}.$$

This clearly implies that $\alpha + \beta = 0$ and $\alpha = 0$. So $\alpha = \beta = 0$.

Thus we have shown that $\alpha\underset{\to}{b} + \beta\underset{\to}{i} = \underset{\to}{0}$ can hold only if $\alpha = \beta = 0$.

Exercise

4. Show that the geometric vectors $\underset{\to}{a}, \underset{\to}{b}, \underset{\to}{c}$ of Example 1 form a linearly dependent set.

Solution

4. We must look for scalars α, β, γ which satisfy the equation

$$\alpha\underset{\to}{a} + \beta\underset{\to}{b} + \gamma\underset{\to}{c} = \underset{\to}{0}.$$

In terms of the geometric vectors $\underset{\to}{i}$ and $\underset{\to}{j}$ we have

$$\alpha(\underset{\to}{i} + 2\underset{\to}{j}) + \beta(\underset{\to}{i} + \underset{\to}{j}) + \gamma(2\underset{\to}{i}) = 0$$

$$\Rightarrow \qquad (\alpha + \beta + 2\gamma)\underset{\to}{i} + (2\alpha + \beta)\underset{\to}{j} = \underset{\to}{0}$$

Hence $\alpha + \beta + 2\gamma = 0$ and $2\alpha + \beta = 0$, and these equations have many non-trivial solutions, such as $\alpha = 2, \beta = -4, \gamma = 1$. Thus

$$2\underset{\to}{a} - 4\underset{\to}{b} + \underset{\to}{c} = \underset{\to}{0},$$

and this set of three geometric vectors is linearly dependent.

The following table summarizes the properties of the sets of geometric vectors studied in the foregoing examples and exercises.

Set	*spans*	*linearly independent*
$\{\underset{\to}{a}, \underset{\to}{b}, \underset{\to}{c}\}$	$\surd$	$\times$
$\{\underset{\to}{b}, \underset{\to}{i}\}$	$\surd$	$\surd$
$\{\underset{\to}{b}, \underset{\to}{d}\}$	$\times$	$\times$
$\{\underset{\to}{i}, \underset{\to}{j}\}$	$\surd$	$\surd$

0.2.5 Basis Vectors

We have expended some considerable effort in determining whether given subsets of geometric vectors in the plane (a) span all the geometric vectors, and (b) are linearly independent. The spanning property is clearly useful—it tells us whether an arbitrarily chosen geometric vector can be expressed a linear combination of the members of our chosen set (which we hope is small)—but how does linear independence help us? The answer is as follows. If a set spans, but is linearly dependent, then each geometric vector may be expressed in *many different* linear combinations of the members of the chosen set. Linear independence guarantees uniqueness—there is only one linear combination of a linearly independent set that equals a given geometric vector.

This result will be proved formally in sub-section 0.3.2.

When we have a subset of the geometric vectors which is both *linearly independent* and *spans the whole set*, we call the subset a basis (*pl.* bases) for the set of geometric vectors. The set $\{\underset{\sim}{i}, \underset{\sim}{j}\}$ clearly is a basis, and in fact constitutes the motivation for considering bases. A basis for the set of geometric vectors has the property that it can be used to establish a grid for (oblique) coordinates of the plane. You should check that the set $\{\underset{\sim}{i}, \underset{\sim}{b}\}$ discussed previously satisfies the definition of a basis.

Example

We have seen that the geometric vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ are linearly dependent. However, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly independent, for if $\alpha\underset{\sim}{a} + \beta\underset{\sim}{b} = \underset{\sim}{0}$, then

$$\alpha(\underset{\sim}{i} + 2\underset{\sim}{j}) + \beta(\underset{\sim}{i} + \underset{\sim}{j}) = 0.$$

This requires that $\alpha + \beta = 0$ and $2\alpha + \beta = 0$. The only solution of this pair of equations is $\alpha = \beta = 0$.

Further, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ span the set of planar geometric vectors for we have, for any $\lambda, \mu \in R$,

$$\lambda\underset{\sim}{i} + \mu\underset{\sim}{j} = (\mu - \lambda)\underset{\sim}{a} + (2\lambda - \mu)\underset{\sim}{b}.$$

Thus $\{\underset{\sim}{a}, \underset{\sim}{b}\}$ is a basis for the set of planar geometric vectors.

Exercises

1. If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly dependent, determine whether

   (i) $\{3\underset{\sim}{a}, 4\underset{\sim}{b}\}$

   (ii) $\{\underset{\sim}{a} + \underset{\sim}{b}, \underset{\sim}{a} - \underset{\sim}{b}\}$

   are linearly dependent sets.

2. If $\{\underset{\sim}{a}, \underset{\sim}{b}\}$ forms a basis, show that

   (i) $\{3\underset{\sim}{a}, 4\underset{\sim}{b}\}$

   (ii) $\{\underset{\sim}{a} + \underset{\sim}{b}, \underset{\sim}{a} - \underset{\sim}{b}\}$

   form bases.

Solutions

1. If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly dependent, then we know that there are numbers α and β, not both zero, such that

   $$\alpha\underset{\sim}{a} + \beta\underset{\sim}{b} = \underset{\sim}{0}.$$

   (i) It follows that

   $$\frac{\alpha}{3}(3\underset{\sim}{a}) + \frac{\beta}{4}(4\underset{\sim}{b}) = 0$$

   and hence $3\underset{\sim}{a}$ and $4\underset{\sim}{b}$ are also linearly dependent.

(ii) $\underset{\to}{a} + \underset{\to}{b}$ and $\underset{\to}{a} - \underset{\to}{b}$ are linearly dependent if we can find numbers λ and μ, not both zero, such that

$$\lambda(\underset{\to}{a} + \underset{\to}{b}) + \mu(\underset{\to}{a} - \underset{\to}{b}) = 0,$$

i.e. such that

$$(\lambda + \mu)\underset{\to}{a} + (\lambda - \mu)\underset{\to}{b} = \underset{\to}{0}.$$

Since $\underset{\to}{a}$ and $\underset{\to}{b}$ are linearly dependent, there are numbers α and β, not both zero, such that

$$\alpha\underset{\to}{a} + \beta\underset{\to}{b} = \underset{\to}{0}.$$

Suppose that we choose $\lambda + \mu = \alpha$

and $\lambda - \mu = \beta$

so that

$$\lambda = \frac{\alpha + \beta}{2} \quad \text{and} \quad \mu = \frac{\alpha - \beta}{2}.$$

Then, if λ and μ are not both zero, we have shown that $\underset{\to}{a} + \underset{\to}{b}$ and $\underset{\to}{a} - \underset{\to}{b}$ are linearly dependent. But this follows at once, since $\lambda = \mu = 0$ implies $\alpha = \beta = 0$, which we know to be false.

2. (i) Suppose that we can find scalars α and β such that

$$\alpha(3\underset{\to}{a}) + \beta(4\underset{\to}{b}) = \underset{\to}{0},$$

i.e. $$3\alpha(\underset{\to}{a}) + 4\beta(\underset{\to}{b}) = \underset{\to}{0}.$$

Since $\{\underset{\to}{a}, \underset{\to}{b}\}$ is linearly independent, it follows that $3\alpha = 4\beta = 0$, which implies that

$$\alpha = \beta = 0.$$

So $\{3\underset{\to}{a}, 4\underset{\to}{b}\}$ is linearly independent.

There remains to be shown that any geometric vector (in the plane), $\underset{\to}{v}$ say, can be expressed as a linear combination

$$\underset{\to}{v} = \lambda(3\underset{\to}{a}) + \mu(4\underset{\to}{b})$$

for some real λ, μ. Since $\{\underset{\to}{a}, \underset{\to}{b}\}$ is a basis we can certainly find real numbers α, β such that

$$\begin{aligned}\underset{\to}{v} &= \alpha\underset{\to}{a} + \beta\underset{\to}{b} \\ &= \frac{\alpha}{3}(3\underset{\to}{\alpha}) + \frac{\beta}{4}(4\underset{\to}{b}),\end{aligned}$$

i.e. $$\lambda = \frac{\alpha}{3} \quad \text{and} \quad \mu = \frac{\beta}{4}.$$

Thus $\{3\underset{\to}{a}, 4\underset{\to}{b}\}$ satisfies both of the conditions for a basis.

(ii) To show that $\{\underset{\to}{a} + \underset{\to}{b}, \underset{\to}{a} - \underset{\to}{b}\}$ is a basis, we must show that it is linearly independent and spans the set of geometric vectors.

Suppose firstly that $\alpha, \beta \in R$ are scalars such that

$$\alpha(\underset{\to}{a} + \underset{\to}{b}) + \beta(\underset{\to}{a} - \underset{\to}{b}) = \underset{\to}{0}.$$

We must show that $\alpha = \beta = 0$. Now we can rewrite the above equation as

$$(\alpha + \beta)\underset{\to}{a} + (\alpha - \beta)\underset{\to}{b} = \underset{\to}{0};$$

since $\{\underset{\to}{a}, \underset{\to}{b}\}$ is a basis it is linearly independent, and so

$$\alpha + \beta = \alpha - \beta = 0.$$

The only solution of these equations is $\alpha = \beta = 0$.

Next we must show that given $\underset{\sim}{v}$ we can find scalars $\lambda, \mu \in R$ such that

$$\underset{\sim}{v} = \lambda(\underset{\sim}{a} + \underset{\sim}{b}) + \mu(\underset{\sim}{a} - \underset{\sim}{b})$$
$$= (\lambda + \mu)\underset{\sim}{a} + (\lambda - \mu)\underset{\sim}{b}.$$

We can certainly find α, β such that

$$\underset{\sim}{v} = \alpha\underset{\sim}{a} + \beta\underset{\sim}{b}$$

since $\{\underset{\sim}{a}, \underset{\sim}{b}\}$ is a basis and so spans the set of geometric vectors. We shall therefore solve our problem if we can find $\lambda, \mu \in R$ such that

$$\lambda + \mu = \alpha$$
$$\lambda - \mu = \beta.$$

Clearly

$$\lambda = \tfrac{1}{2}(\alpha + \beta), \qquad \mu = \tfrac{1}{2}(\alpha - \beta)$$

will do the job required.

0.2.6 Summary of Section 0.2

In this section we have defined the terms

geometric vector	(page 11)
resultant	(page 13)
addition (of geometric vectors)	(page 13)
scalar multiplication (of geometric vectors)	(page 16)
scalar	(page 16)
linear combination	(page 18)
span	(page 18)
linearly dependent	(page 20)
linearly independent	(page 20)
basis (for geometric vectors in the plane)	(page 21)

We introduced the notation

$\underset{\rightarrow}{AB}$	(page 11)
$\underset{\sim}{t}$	(page 13)
$\underset{\sim}{0}$	(page 15)
$-\underset{\sim}{t}$	(page 15)
$\underset{\sim}{a} - \underset{\sim}{b}$	(page 15)
$\Rightarrow$	(page 15)

Techniques

1. Addition of geometric vectors.
2. Scalar multiplication of geometric vectors.
3. Determine whether a given set of geometric vectors spans or is linearly independent.

0.3 VECTOR SPACES

0.3.1 The Algebra of Lists

How can we best make use of a basis? Let us begin by considering the basis $\{\underset{\rightarrow}{i}, \underset{\rightarrow}{j}\}$ consisting of the 'unit' geometric vectors in the directions of the x-axis and y-axis respectively. If $\underset{\rightarrow}{a}$ and $\underset{\rightarrow}{b}$ are any two geometric vectors in the plane then there are scalars (real numbers) α_1, α_2, β_1, β_2 such that

$$\underset{\rightarrow}{a} = \alpha_1 \underset{\rightarrow}{i} + \alpha_2 \underset{\rightarrow}{j}$$
$$\underset{\rightarrow}{b} = \beta_1 \underset{\rightarrow}{i} + \beta_2 \underset{\rightarrow}{j}.$$

Then

$$\begin{aligned} \underset{\rightarrow}{a} + \underset{\rightarrow}{b} &= (\alpha_1 \underset{\rightarrow}{i} + \alpha_2 \underset{\rightarrow}{j}) + (\beta_1 \underset{\rightarrow}{i} + \beta_2 \underset{\rightarrow}{j}) \\ &= (\alpha_1 \underset{\rightarrow}{i} + \beta_1 \underset{\rightarrow}{i}) + (\alpha_2 \underset{\rightarrow}{j} + \beta_2 \underset{\rightarrow}{j}) \quad \text{using properties } A2 \text{ and } A5 \\ &= (\alpha_1 + \beta_1)\underset{\rightarrow}{i} + (\alpha_2 + \beta_2)\underset{\rightarrow}{j} \qquad \text{using property } B4. \end{aligned}$$

The remarkable thing to notice is that addition of the geometric vectors $\underset{\rightarrow}{a}$ and $\underset{\rightarrow}{b}$ is accomplished by adding separately the scalars multiplying $\underset{\rightarrow}{i}$ and the scalars multiplying $\underset{\rightarrow}{j}$. That is to say, having chosen $\{\underset{\rightarrow}{i}, \underset{\rightarrow}{j}\}$ as the basis in terms of which our geometric vectors shall be expressed, addition of geometric vectors is reduced to addition in R, twice. This feature is most easily expressed in terms of the addition of 2×1 column matrices.

In the context of matrices, "2×1" is read as "two by one"; it refers to a matrix with two rows and one column.

The matrix equation

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \end{bmatrix}$$

contains all the information to determine $\underset{\rightarrow}{a} + \underset{\rightarrow}{b}$ given $\underset{\rightarrow}{a}$ and $\underset{\rightarrow}{b}$ in terms of $\{\underset{\rightarrow}{i}, \underset{\rightarrow}{j}\}$. Each geometric vector is represented by a column matrix or list whose first entry is the coefficient of $\underset{\rightarrow}{i}$ and whose second entry is the coefficient of $\underset{\rightarrow}{j}$. It is no accident that addition of lists corresponds to addition of geometric vectors; indeed it would not take you very long to show that (matrix) addition of lists has essentially the properties $A1 - A5$ of geometric vectors.

Each real number in a matrix is called an *element* or *entry* of the matrix.

In the same way scalar multiplication of a geometric vector gives us

$$\begin{aligned} \lambda \underset{\rightarrow}{a} &= \lambda(\alpha_1 \underset{\rightarrow}{i} + \alpha_2 \underset{\rightarrow}{j}) \\ &= \lambda(\alpha_1 \underset{\rightarrow}{i}) + \lambda(\alpha_2 \underset{\rightarrow}{j}) \qquad \text{property } B3 \\ &= (\lambda\alpha_1)\underset{\rightarrow}{i} + (\lambda\alpha_2)\underset{\rightarrow}{j} \qquad \text{property } B2, \end{aligned}$$

which corresponds to the matrix equation

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \lambda\alpha_1 \\ \lambda\alpha_2 \end{bmatrix}.$$

Check the matrix multiplication!

Here we have used the matrix corresponding to a dilation or scaling, which you first met in the Foundation Course, to represent scalar multiplication of geometric vectors, which corresponds to the scaling of a translation.

M101 Block IV *Unit 3*, Sections 3.4 and 3.5.

Since the scalings which concern us here are uniform (the same for each axis) we can condense the matrix equation above to

$$\lambda\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \lambda\alpha_1 \\ \lambda\alpha_2 \end{bmatrix}.$$

The rules we now have for addition and scalar multiplication of lists look deceptively like the corresponding rules defined for R^2 in sub-section 0.1.1. We exploited this correspondence in the Foundation Course, where we cheerfully rewrote the coordinates of a point in the plane, that is, an element of R^2, as a matrix (or list). If we want to express this representation formally we should say that the two number-pairs $(1, 0)$ and $(0, 1)$ have the requisite

properties to act as a basis for R^2; every member $(\lambda, \mu) \in R^2$ can be expressed uniquely as

$$(\lambda, \mu) = \lambda(1,0) + \mu(0,1).$$

The basis $\{(1,0),(0,1)\}$ now determines the list of coefficients $\begin{bmatrix} \lambda \\ \mu \end{bmatrix}$ to represent the pair (λ, μ). The full power of the matrix representation will not become apparent until we find it necessary to use a different set of pairs as a basis for R^2, in the same way as we found alternative bases for the set of geometric vectors in the plane.

We can extend our discussion of geometric vectors by considering translations in three-dimensional space. In this case no two geometric vectors can be found which span the whole set; we need three to do the job. Given a frame of Cartesian coordinate axes for space, we can choose $\{\underset{\to}{i}, \underset{\to}{j}, \underset{\to}{k}\}$ to be 'unit' geometric vectors in the direction of the x, y and z axes. Then an arbitrarily chosen geometric vector $\underset{\to}{a}$ can be expressed as a linear combination

$$\underset{\to}{a} = \alpha_1 \underset{\to}{i} + \alpha_2 \underset{\to}{j} + \alpha_3 \underset{\to}{k}$$

for some $\alpha_1, \alpha_2, \alpha_3 \in R$ and, likewise, for any $\underset{\to}{b}$ we can find scalars β_1, β_2, $\beta_3 \in R$ such that

$$\underset{\to}{b} = \beta_1 \underset{\to}{i} + \beta_2 \underset{\to}{j} + \beta_3 \underset{\to}{k}.$$

Addition and scalar multiplication of geometric vectors in three dimensions satisfy all the properties listed in Section 0.2 and so we can show that

$$\underset{\to}{a} + \underset{\to}{b} = (\alpha_1 + \beta_1)\underset{\to}{i} + (\alpha_2 + \beta_2)\underset{\to}{j} + (\alpha_3 + \beta_3)\underset{\to}{k}$$

and

$$\lambda \underset{\to}{a} = (\lambda\alpha_1)\underset{\to}{i} + (\lambda\alpha_2)\underset{\to}{j} + (\lambda\alpha_3)\underset{\to}{k}.$$

The details are similar to the two-dimensional case, and we shall not give them here.

This leads us to define an algebra of *lists with three entries* (3×1 matrices) by

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \end{bmatrix}$$ Equation (1)

and

$$\lambda \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \lambda\alpha_1 \\ \lambda\alpha_2 \\ \lambda\alpha_3 \end{bmatrix}$$ Equation (2)

These lists give a neat way of specifying geometric vectors; but do they only give us an alternative notation, or do they suggest anything new? Let's forget for a moment the origins of the lists. Equations (1) and (2) define ways of manipulating lists of numbers. There is no reason why we should always have only two or three elements in the list. Equations (1) and (2) can be extended to lists with more than three elements; for example, we can write

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \\ \cdot \\ \cdot \\ \cdot \\ a_n + b_n \end{bmatrix}$$

and

$$\lambda \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \\ \cdot \\ \cdot \\ \cdot \\ \lambda a_n \end{bmatrix}.$$

But this is rather futile if we have a physical or mathematical interpretation only when the lists contain three elements or fewer. However, we can use these lists to describe situations other than the algebra of geometric vectors, and we can interpret results and concepts in one situation (for example, basis and linear independence) to give results and concepts in another. That being so, we shall go on to discuss the *abstract* structure which typifies all the exemplary situations.

What else can we represent by lists?

Example 1 Polynomial Functions

Consider the set of all polynomial functions of the form

$$p: x \longmapsto ax^3 + bx^2 + cx + d \qquad (x \in R)$$

where a, b, c and d are real numbers.

Remember that a, b, c and d can be *any* real numbers, and so a function such as $f: x \longmapsto 0x^3 + 0x^2 + 1$ is included in this set of functions.

We can represent p by the four coefficients a, b, c and d, which we can arrange as a list:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

The addition of two such polynomial functions corresponds to the addition of the corresponding two lists. Thus if

$$p_1: x \longmapsto a_1x^3 + b_1x^2 + c_1x + d_1 \qquad (x \in R)$$

and

$$p_2: x \longmapsto a_2x^3 + b_2x^2 + c_2x + d_2 \qquad (x \in R)$$

then we *define* addition of functions by

$$p_1 \dot{+} p_2: x \longmapsto p_1(x) + p_2(x) \qquad (x \in R)$$

and the scalar multiplication of functions by

$$\lambda p: x \longmapsto \lambda p(x) \qquad (x \in R).$$

It does not take much effort to see that $p_1 + p_2$ corresponds to the list

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{bmatrix}$$

and the function λp corresponds to the list

$$\lambda \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \\ \lambda c \\ \lambda d \end{bmatrix}.$$

By considering polynomials of degree higher than three we would get examples of lists with more than four elements. It is worth noting in this

example that although the definitions of addition and scalar multiplication of functions are 'obvious', we are considering functions as the members of a *set on which operations can be defined.*

Example 2 Solutions of Differential Equations

Often in applied mathematics we are faced with the problem of finding a function, f say, which is related to a given function g through its derivatives. For example, we may need to determine the set of functions

$$\{f: f''(x) - 3f'(x) + 2f(x) = g(x) \quad \text{for all } x \in R\}.$$

We call an equation such as the one above a *differential equation*, and each f in the stated set *satisfies the equation* or is a *solution of the equation.*

Differential equations were introduced in M101 Block V *Unit 2.*

Suppose, for example, that g were the zero function, defined by

$$g: x \longmapsto 0 \qquad \text{for } x \in R.$$

Then the function

$$f_1: x \longmapsto e^x \qquad \text{for } x \in R$$

is one function which satisfies the equation.

In fact,

$$f_1'(x) = e^x \quad \text{and} \quad f_1''(x) = e^x$$

so that

$$f_1''(x) - 3f_1'(x) + 2f_1(x) = e^x - 3e^x + 2e^x = 0.$$

Another function which satisfies the equation is

$$f_2: x \longmapsto e^{2x} \qquad \text{for } x \in R$$

We shall show later in this course that *any* solution of this differential equation has the form

$$\alpha f_1 + \beta f_2$$

where α and β are real numbers, and f_1 and f_2 are the functions given above. If we take f_1 and f_2 as *basic solutions*, then any solution of the form $\alpha f_1 + \beta f_2$ can be represented by the list $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. The particular solutions f_1 and f_2 can be represented by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively, and in general the list $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ represents the function

$$x \longmapsto \alpha e^x + \beta e^{2x} \qquad \text{for } x \in R.$$

You may like to verify that the lists

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha + \gamma \\ \beta + \delta \end{bmatrix}$$

and

$$\lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \lambda\alpha \\ \lambda\beta \end{bmatrix}$$

also represent solutions of the equation.

Exercise

Show that the function

$$x \longmapsto \alpha e^x + \beta e^{2x} \qquad \text{for } x \in R$$

satisfies the differential equation

$$f''(x) - 3f'(x) + 2f(x) = 0.$$

Solution

If $\quad f(x) = \alpha e^x + \beta e^{2x}$

then $f'(x) = \alpha e^x + 2\beta e^{2x}$

and $f''(x) = \alpha e^x + 4\beta e^{2x}$

So $\quad f''(x) - 3f'(x) + 2f(x) = \alpha e^x + 4\beta e^{2x} - 3\alpha e^x - 6\beta e^{2x} + 2\alpha e^x + 2\beta e^{2x}$

$= 0 \qquad$ for all $x \in R$.

0.3.2 Vector Spaces

From what we have learned about geometric vectors, we are now able to construct an abstract mathematical structure called a *vector space.*

A feature common to the geometric vectors and the examples in the last sub-section is that, in each case, we had a set on which we could sensibly define *addition* and *multiplication by a scalar.*

We shall take the structure which we have developed on the set of geometric vectors as our model, and discuss an arbitrary set with operations called *addition* and *multiplication by a scalar* defined on it. If the structure satisfies the following axioms, then we call it a vector space, and we call its elements *vectors.*

The set of geometric vectors is a particular example of a vector space, and it is the origin of the subject in geometry which motivates this use of the word *space.*

One of the purposes of talking about structures such as vector spaces in the abstract is that we hope to be able to represent a number of apparently different structures in the same terms. Thus when we refer to a vector in a vector space, it may be a number pair, a geometric vector, a solution to a differential equation, a list, a polynomial function, or one of many other things. By proving theorems about vector spaces in general, we are able to obtain results for all these different situations at once. This 'increase in productivity' is a prime reason for generalization in mathematics.

We choose a notation which is not too suggestive of any one particular example, and use boldface letters such as **v**, **a**, **i** to represent vectors.

In order that a set V should be called a vector space, we require that the operations of *addition* of members of V and *scalar multiplication* of members of V should be defined and have the following properties (modelled upon those of geometric vectors given in Section 0.2):

Axioms of a Vector Space

For any elements $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of V and any real numbers α, β:

A1 $\quad \mathbf{v}_1 + \mathbf{v}_2$ is a unique element of V (V is closed for addition)
A2 $\quad \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3$ (addition is associative)
A3 $\quad$ There is an element in V, which we call $\mathbf{v}_0$ such that

$$\mathbf{v} + \mathbf{v}_0 = \mathbf{v}$$

A4 $\quad$ For each $\mathbf{v}$ there is an element $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{v}_0$.
A5 $\quad \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$ (addition is commutative)
B1 $\quad \alpha\mathbf{v}$ is an element of V
B2 $\quad (\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$
B3 $\quad \alpha(\mathbf{v}_1 + \mathbf{v}_2) = (\alpha\mathbf{v}_1) + (\alpha\mathbf{v}_2)$
B4 $\quad (\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.
B5 $\quad 1 \times \mathbf{v} = \mathbf{v}$

These ten axioms are the axioms of a vector space. There are two important points to note. Strictly speaking, we should call V a vector space over the real numbers or a real vector space, because vector spaces exist involving sets of scalars other than the set of real numbers; we shall discuss only vector spaces over the real numbers. Secondly, we have taken as implicit all the relevant properties of the real numbers, and these should really be stated along with the other axioms. Any other set with these properties can be taken as the set of scalars in place of the set of real numbers to give a different vector space.

The axioms of a vector space therefore consist of three sets of axioms:

(i) those applying to the set of vectors only (*A1* to *A5* above);
(ii) those applying to the set of scalars only (not stated above: the missing axioms are the axioms of what is known in mathematics as a *field*);
(iii) those which describe the interaction between the set of scalars and the set of vectors (*B1* to *B5* above).

We define an operation of *subtraction* of vectors by

$$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_1 + (-\mathbf{v}_2).$$

From axiom *A4*, it follows that $\mathbf{v} - \mathbf{v} = \mathbf{v}_0$.

The Zero Vector

In a vector space, an element $\mathbf{v}_0$ which satisfies axiom *A3* is called a *zero vector*. It follows from the axioms that in any vector space V there is only one zero vector. For suppose there are two vectors $\mathbf{v}_0$ and $\mathbf{v}_0'$ which satisfy axiom *A3*. That is,

$$\mathbf{v} + \mathbf{v}_0 = \mathbf{v}$$

$$\mathbf{v} + \mathbf{v}_0' = \mathbf{v},$$

where, in each equation, $\mathbf{v}$ is any element of V. Let us put $\mathbf{v} = \mathbf{v}_0'$ in the first equation and $\mathbf{v} = \mathbf{v}_0$ in the second equation. We obtain

$$\mathbf{v}_0' + \mathbf{v}_0 = \mathbf{v}_0'$$

$$\mathbf{v}_0 + \mathbf{v}_0' = \mathbf{v}_0.$$

By axiom *A5*,

$$\mathbf{v}_0' + \mathbf{v}_0 = \mathbf{v}_0 + \mathbf{v}_0'$$

i.e. $\quad \mathbf{v}_0' = \mathbf{v}_0,$

so the zero vector is unique.

Since the zero vector in a vector space behaves just like the zero geometric vector, we shall call this element *the* zero vector and denote it by $\mathbf{0}$, just as we had $\underset{\rightarrow}{0}$ for the zero geometric vector. (In terms of lists, $\mathbf{0}$ is the list in which every entry is zero.)

Further properties of $\mathbf{0}$ can be deduced from the axioms. For example, it can be shown that

$$\alpha\mathbf{0} = \mathbf{0},$$

where α is any real number.

Exercises

1. (i) Which of the examples of sub-section 0.3.1 describe vector spaces?
   (ii) The set of all polynomial functions of degree n with the operations of addition of functions and multiplication of a function by a real number is not a vector space. Why not?

   Suggest a suitable modification to make it a vector space.

HINT: A (real) polynomial function of degree n is a function of the form

$$x \longmapsto a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

for $x \in R$ in which a_i are real numbers ($i = 0, 1, \ldots, n$) and $a_n \neq 0$.

2. In each of the following cases state whether the given set of lists forms a vector space for the operations of addition of lists and multiplication of a list by a scalar. In each case give reasons for your answer.

(i) The set of all lists $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where x_1, x_2 are positive real numbers.

(ii) The set of all lists $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where x_1, x_2 are real numbers and $x_1 + x_2 = 0$.

(iii) The set of all lists $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where x_1 and x_2 are real numbers and $x_1 < x_2$.

(iv) The set of all lists $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, where x_1, x_2 and x_3 are real numbers such that the function

$$f : t \longmapsto x_1 t^2 + x_2 t + x_3 \qquad \text{for } t \in R$$

satisfies $f(k) = 0$, where k is a fixed real number.

3. If V is a vector space with zero vector $\mathbf{0}$, show that $\{\mathbf{0}\}$ is also a vector space.

Solutions

1. (i) Both examples describe vector spaces. (Example 1 is discussed by implication below.)

   (ii) The problem is caused when we add, say, the polynomial function $x \longmapsto -x^n$ to the polynomial function $x \longmapsto x^n + x^{n-1}$. Both are of degree n, but their sum is the polynomial $x \longmapsto x^{n-1}$, which is of degree $n - 1$, so $+$ is not closed, i.e. axiom $A1$ is violated. A simple modification is to consider the set of polynomials of degree *less than or equal to n*. With the suggested operations, this set is indeed a vector space.

2. (i) No. For example, multiplication by a negative scalar takes us out of the set, i.e. axiom $B1$ is violated.

   (ii) Yes. All the axioms are satisfied. (In fact, all the points in R^2 corresponding to the vectors lie on the line defined by the equation $y + x = 0$.)

   (iii) No. For example, if $x_1 < x_2$ and $\alpha < 0$, then $\alpha x_1 > \alpha x_2$, i.e. $\alpha\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ does not belong to the given set, so axiom $B1$ is violated.

   (iv) Yes. All the axioms of a vector space are satisfied. (Each function has a graph which passes through the fixed point $(k, 0)$.)

3. We check that the axioms of a vector space are satisfied.

   A1 $\mathbf{0} + \mathbf{0} = \mathbf{0}$, since $\mathbf{0}$ is the zero element of V, so $\{\mathbf{0}\}$ is closed for addition.
   A3 $\mathbf{0} \in \{\mathbf{0}\}$
   A4 $-\mathbf{0} = \mathbf{0}$, since $\mathbf{0} + \mathbf{0} = \mathbf{0}$.
   B1 $\alpha\mathbf{0} = \mathbf{0} \in \{\mathbf{0}\}$ (by a result quoted in the text).

   The other axioms are automatically satisfied, since they are satisfied for all elements of V, and $\mathbf{0} \in V$.

   Hence $\{\mathbf{0}\}$ is a (real) vector space.

Where next?

In the case of geometric vectors, we introduced the idea of a basis. The development of this idea depended on the concepts of linear combination of vectors and linear dependence. We can extend these ideas to the more general concept of a vector space.

We also made passing reference to these ideas in our examples in sub-section 0.3.1. In the differential equation example, we shall see that every solution of

$$f''(x) - 3f'(x) + 2f(x) = 0$$

can be represented in terms of *two* basic solutions, for example

$$f_1 : x \longmapsto e^x \qquad \text{for } x \in R$$

$$f_2 : x \longmapsto e^{2x} \qquad \text{for } x \in R.$$

How do we choose a basis for a vector space? How many vectors do we need? If we can settle the question of how many vectors we need—can we select that number of vectors at random?

Before we extend our idea of a *basis* to an abstract vector space, we shall define linear dependence in this context.

Linear Dependence and Independence

The following definitions generalize the notion of linear dependence which we introduced for geometric vectors.

If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ are vectors from a vector space, then an expression of the form

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots + \alpha_n\mathbf{v}_n,$$

where the α_i are real numbers, is called a *linear combination* of vectors.

The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *linearly dependent* if and only if there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, which are not all zero, such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0};$$

in other words, if $\mathbf{0}$ is a non-trivial linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

A set of vectors which is not linearly dependent is said to be *linearly independent*. We can define this term in a more positive way as follows.

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is linearly independent if and only if the equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

has just one solution, namely

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0.$$

Remember that we use the terms *dependent* and *independent* in this way because we can express some members of a linearly dependent set in terms

of the others. For example, if α_1 is not zero, we can use the axioms of a vector space to write

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

in the form

$$\alpha_1\mathbf{v}_1 = (-\alpha_2)\mathbf{v}_2 + (-\alpha_3)\mathbf{v}_3 + \cdots + (-\alpha_n)\mathbf{v}_n,$$

and then divide by α_1 to give:

$$\mathbf{v}_1 = \frac{-\alpha_2}{\alpha_1}\mathbf{v}_2 + \frac{-\alpha_3}{\alpha_1}\mathbf{v}_3 + \cdots + \frac{-\alpha_n}{\alpha_1}\mathbf{v}_n$$

i.e. $\mathbf{v}_1$ depends on (i.e. is a linear combination of) the other vectors. In general, if a set of vectors is linearly dependent, some of the vectors in the set (not necessarily every vector, because some of the α's may be zero) can be expressed in terms of the others. In other words, some of the elements in the set are redundant.

Exercises

4. In each of the following parts a set of vectors is given. In each case state whether or not the set is linearly independent.

   In those cases where the set is linearly dependent, express one of the vectors in the set as a linear combination of the others.

   (i) The vector space R^3 is defined as the set of ordered triples of real numbers, i.e., $\{(x, y, z): x, y, z \in R\}$ with *componentwise* addition and scalar multiplication,

   $$(x, y, z) + (u, v, w) = (x + u, y + v, z + w)$$

   $$\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z).$$

   (a) $\{(1, -1, 0), (0, 1, 0), (1, 0, 0)\}$

   (b) $\{(2, 0, 0), (0, 3, 0), (0, 0, 5)\}$

   The zero vector in R^3 is the triple $(0, 0, 0)$.

   (ii) The set of functions $\{f, g\}$, where

   $$f: x \longmapsto x \qquad (x \in R)$$

   $$g: x \longmapsto x^2 \qquad (x \in R),$$

   with the operations of addition of functions and multiplication of a function by a real number..

   The zero vector in this case is the function

   $$0: x \longmapsto 0 \quad (x \in R).$$

5. If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is linearly independent, show that if

   $$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_n\mathbf{v}_n$$

   then

   $$\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_n = \beta_n.$$

   Notice that this result implies that a vector $\mathbf{v}$ cannot be expressed in two different ways as a linear combination of a set of linearly independent vectors.

6. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors, prove that any subset of this set is also linearly independent.

7. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly dependent subset of a vector space V, prove that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{w}\}$ is also linearly dependent, where $\mathbf{w}$ is any element in V.

Solutions

4. (i) (a) This set of triples is linearly dependent: for instance,

$$(1,0,0) = (1,-1,0) + (0,1,0).$$

(b) This set of triples is linearly independent. If

$$\alpha_1(2,0,0) + \alpha_2(0,3,0) + \alpha_3(0,0,5) = (0,0,0)$$

then $\quad (2\alpha_1, 3\alpha_2, 5\alpha_3) = (0,0,0),$

whence $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

(ii) The set of functions is linearly independent, because $\alpha f + \beta g = 0$ implies that

$$\alpha x + \beta x^2 = 0 \text{ for all values of } x,$$

Try $x = 1$ and $x = -1$.

and this is possible only if $\alpha = \beta = 0$.

5. Using the axioms of a vector space, we can show that the given equation is equivalent to

$$(\alpha_1 - \beta_1)\mathbf{v}_1 + (\alpha_2 - \beta_2)\mathbf{v}_2 + \cdots + (\alpha_n - \beta_n)\mathbf{v}_n = \mathbf{0}.$$

Since the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent, the coefficients of the vectors in the above equation are all zero, so

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \cdots = \alpha_n - \beta_n = 0,$$

which proves the required result.

6. Suppose, in contradiction to what we want to prove, that the subset $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent: then there are numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ (not all zero) such that

This popular method of proof is known as *proof by contradiction*. We suppose that the stated result is *false*, and show that this supposition leads to a contradiction in terms of the given hypothesis. We thereby deduce that the stated result is *true*.

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = \mathbf{0}.$$

Therefore

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k + 0\mathbf{v}_{k+1} + \cdots + 0\mathbf{v}_n = \mathbf{0}$$

But not all the $\alpha_1, \ldots, \alpha_n$ are zero, and hence the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent—which is a contradiction.

7. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent, then there are numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}.$$

Hence

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n + 0\mathbf{w} = \mathbf{0}$$

Not all the coefficients in this last equation are zero, and so we have proved the required result.

0.3.3 Bases and Dimension

In sub-section 0.2.4 we saw that it is possible to select two geometric vectors in a plane, and then to specify every geometric vector in the plane as a linear combination of those two. Similarly, in three dimensions we need to select three geometric vectors. We called such a set a basis, and we now wish to extend the same idea to an abstract vector space.

The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is said to *span* the vector space V if for each element $\mathbf{w}$ in V we can find scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$, such that

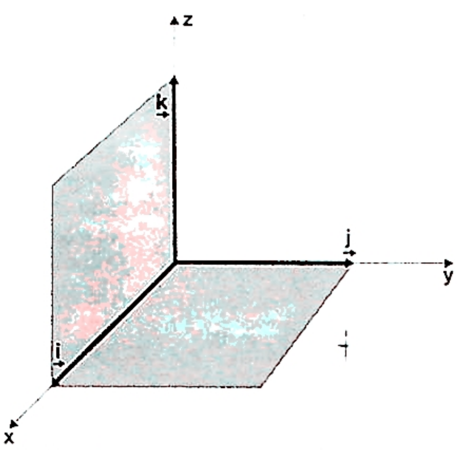
$$\mathbf{w} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots + \alpha_m\mathbf{v}_m.$$

If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent *and* spans the vector space V, then we say that it forms a basis for V.

Essentially, a basis contains *the minimum number of elements which are required to span the space.* In Exercise 5 of sub-section 0.3.2 we saw that any vector can be expressed in a *unique* way as a linear combination of the elements of a basis.

For example, the set $\{\underset{\rightarrow}{i}, \underset{\rightarrow}{j}, \underset{\rightarrow}{k}\}$ spans the three-dimensional geometric vector space, because each geometric vector $\underset{\rightarrow}{r}$ can be expressed in the form

$$\underset{\rightarrow}{r} = x\,\underset{\rightarrow}{i} + y\,\underset{\rightarrow}{j} + z\,\underset{\rightarrow}{k}.$$



Here $\underset{\rightarrow}{i}, \underset{\rightarrow}{j}$ and $\underset{\rightarrow}{k}$ play the parts of $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ and we know that it is possible to find the appropriate values x, y and z which play the parts of α_1, α_2 and α_3. Any set of geometric vectors containing $\underset{\rightarrow}{i}, \underset{\rightarrow}{j}$ and $\underset{\rightarrow}{k}$ and other geometric vector(s) would also span the space, but it would not form a basis, since such a set would be linearly dependent (the other geometric vector(s) would be redundant).

Exercise

Show that the set $\{(1,0,0),(1,1,1),(0,0,1)\}$ is a basis for the space R^3 of all triples of real numbers.

Solution

Any triple $(x_1, x_2, x_3) \in R^3$ can be written as

$$(x_1, x_2, x_3) = (x_1 - x_2)(1,0,0) + x_2(1,1,1) + (x_3 - x_2)(0,0,1),$$

so the three given vectors *span* R^3.

Also the set of triples is *linearly independent*, since

$$\alpha_1(1,0,0) + \alpha_2(1,1,1) + \alpha_3(0,0,1) = (0,0,0)$$

implies $(\alpha_1 + \alpha_2, \alpha_2, \alpha_2 + \alpha_3) = (0,0,0)$

i.e.

$$\begin{aligned} \alpha_1 + \alpha_2 &= 0 \\ \alpha_2 &= 0 \\ \alpha_2 + \alpha_3 &= 0, \end{aligned}$$

whence

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

It follows that the given set of triples is a *basis*.

As a result of this last exercise we have two distinct bases, namely $\{(1,0,0),(0,1,0),(0,0,1)\}$ and $\{(1,0,0),(1,1,1),(0,0,1)\}$, for the same vector space R^3, the space of all triples; and in this case both bases consist of three vectors. In fact, although we shall not prove it here, this always happens: for any two sets of basis vectors for the same vector space, there is always the same number of vectors in each basis. This enables us to make the following definition.

The proof will be given in *Unit 1*.

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then we say that the vector space is of *dimension n*.

If it is impossible to find a finite number of elements of a vector space V which form a basis for V, and $V \neq \{\mathbf{0}\}$, then we say that V has *infinite dimension*.

It is in fact also true that any set of n linearly independent vectors in a vector space V of dimension n is a basis for V, but the proof of this result must be deferred until later.

If we assume these results, then we can see that, since $\{(1,0),(0,1)\}$ basis of the vector space R^2 of ordered pairs of real numbers, this vector space therefore has dimension 2. The set $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for R^3, the space of ordered triples, and this vector space is therefore of dimension 3. Let us look now at some non-geometric examples.

Example 1

The set of all polynomial functions of degree 2 or less, i.e. of the form:

$$f: x \longmapsto ax^2 + bx + c \qquad (x \in R)$$

where $a, b, c \in R$, forms a vector space with the operations of addition of functions and multiplication of a function by a real number.

We can find many sets of three vectors in this vector space which are linearly independent. One such set, which is particularly simple, consists of the vectors

$$\begin{aligned} &f_1: x \longmapsto 1 &\qquad (x \in R) \\ &f_2: x \longmapsto x &\qquad (x \in R) \\ &f_3: x \longmapsto x^2 &\qquad (x \in R). \end{aligned}$$

Any quadratic function can be expressed in terms of these three, and hence they form a basis for the vector space. The dimension of the space is therefore 3. The function

$$f: x \longmapsto 3x^2 - 2x + 4 \qquad (x \in R),$$

can be written as a linear combination of the basis vectors:

$$\mathbf{f} = 3\mathbf{f}_3 - 2\mathbf{f}_2 + 4\mathbf{f}_1,$$

i.e.

$$(x \longmapsto 3x^2 - 2x + 4) = 3(x \longmapsto x^2) - 2(x \longmapsto x) + 4(x \longmapsto 1).$$

The f's are shown boldface in order to emphasize the fact that we are considering the functions to be elements of a vector space.

Example 2

In sub-section 0.3.1, we stated that any solution of the equation

$$f''(x) - 3f'(x) + 2f(x) = 0$$

can be expressed in terms of the two solutions

$$\mathbf{f}_1 : x \longmapsto e^x \qquad (x \in R),$$

$$\mathbf{f}_2 : x \longmapsto e^{2x} \qquad (x \in R).$$

In other words, these two functions span the space of solutions. Since the two solutions $\mathbf{f}_1$ and $\mathbf{f}_2$ are linearly independent, the set of all solutions of the equation forms a vector space of dimension 2.

If you are worried by the statement that $\mathbf{f}_1$ and $\mathbf{f}_2$ are linearly indpendent, you might like to try to prove it.

0.3.4 Summary of Section 0.3

In this section we defined the terms

list	(page 24)
addition and scalar multiplication of functions	(page 26)
solution of a differential equation	(page 27)
zero function	(page 27)
vector space	(page 28)
zero vector	(page 29)
linear combination	(page 31)
linear dependence and independence	(page 31)
basis	(page 34)
dimension	(page 35)

We introduced the notation

$$\lambda \begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} \text{ for the list } \begin{bmatrix} \lambda\alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \lambda\alpha_n \end{bmatrix} \qquad \text{(pages 24, 25, 26)}$$

$\mathbf{v}, \mathbf{a}, \mathbf{i}$ (page 28)

$\mathbf{0}$ (page 29)

R^3, the space of ordered triples of real numbers. (page 32)

We have seen how sets of ordered pairs (or triples) and suitable sets of functions may be endowed with the operations of addition and scalar multiplication to yield an algebraic structure with properties analogous to those of addition and scalar multiplication of geometric vectors.

A structure which possesses the selected properties is called a (real) vector space. The concept of a vector space is of fundamental importance; indeed the whole course is built round it. In *Unit 1* we shall put the subject on firm foundations by proving the unproved assertions of this section.

Meanwhile, we shall continue this unit by investigating transformations (functions) between vector spaces which preserve the fundamental linear structure.

0.4 MAPPINGS OF VECTOR SPACES

0.4.0 Introduction

We have seen that vector spaces are interesting mathematical structures which model many different situations. But of itself this does not give us a way of solving problems.

The concept of a function is often a way of introducing greater sophistication into a structure. We shall find that vector spaces become richer and more interesting when we introduce functions from one vector space to another. Such functions are often referred to as *mappings* and it is then usual to say that 'a maps to b' when $a \longmapsto b$.

0.4.1 Mapping One Vector Space to Another

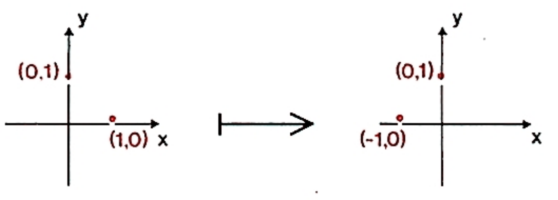
Example 1

Consider the mapping of R^2 to R^2 defined by

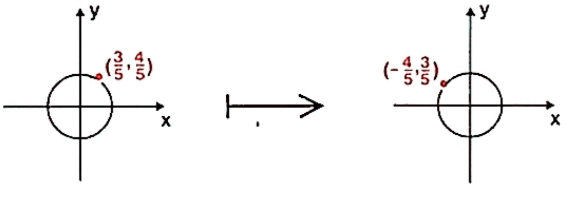
$$(x, y) \longmapsto (-y, x).$$

Let us have a look at what happens to the vectors $(1, 0)$ and $(0, 1)$. We have

$$(1, 0) \longmapsto (0, 1) \quad \text{and} \quad (0, 1) \longmapsto (-1, 0).$$



The mapping has the effect of rotating these vectors through an angle $\pi/2$ anti-clockwise about the origin, and this is indeed the effect on the entire plane.



In this example, any circle centred at the origin maps onto itself. Every point of the set $\{(x, y): x^2 + y^2 = 1\}$ moves but the set itself remains unchanged.

For example, $(\frac{3}{5}, \frac{4}{5}) \longmapsto (-\frac{4}{5}, \frac{3}{5})$.

Now we shall look at a particularly significant example.

Example 2

Consider the mapping of R^2 to R^2 defined by

$$(x, y) \longmapsto (-y, y).$$

To see the effect of this mapping, it is worth looking at what happens to the vectors in a basis. If we choose the simple basis $\{(1, 0), (0, 1)\}$, we have

$$(1, 0) \longmapsto (0, 0)$$

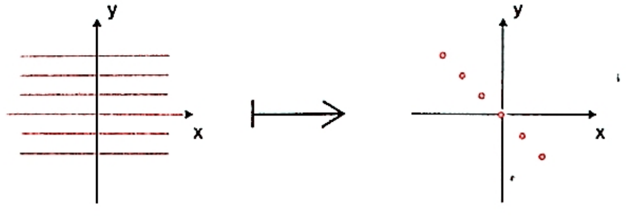
$$(0, 1) \longmapsto (-1, 1).$$

It is helpful to express everything in terms of our chosen basis, so let us write

$$\mathbf{i} = (1, 0) \qquad \mathbf{j} = (0, 1).$$

In terms of the vectors **i** and **j**, we have that **j** maps to $-\mathbf{i} + \mathbf{j}$, but **i** maps to the zero vector.

We shall look at this mapping in a little more detail. What happens to the x-axis? On this axis y is zero and so every point on the x-axis maps to $(0, 0)$: the entire x-axis shrinks into the origin. What about the line $y = 1$? Every point on this line maps to the point $(-1, 1)$.



In fact, the entire plane maps on to the line whose equation is

$$x + y = 0.$$

In terms of the basis vectors, every element in the image set is a scalar multiple of the vector $-\mathbf{i} + \mathbf{j}$.

The image set has dimension 1, and so the effect of the mapping is to "lose" a dimension from our vector space. This is equivalent to saying that this mapping is not one-one. If we start with a point, P say, in the plane and map it to a point Q, on the line $x + y = 0$, then we cannot map back to the original point P. This is because the point Q on the line $x + y = 0$ (of the codomain) corresponds to the *whole* of the line parallel to the x-axis through P.

We have chosen these two examples, in which R^2 was identified with the Cartesian plane, to give you a visualization of the sort of mappings we are going to consider. One of the pleasant features of linear algebra is that by considering a geometric situation we can often throw lights on non-geometric situations (and vice versa). Thus, a non-geometric analogue of the last case, where we "lost" a dimension, is provided by the following example.

Example 3

Let P_4 be the vector space of all polynomial functions of degree 3 or less. The operation of differentiation can be thought of as a "mapping" with domain P_4. Each polynomial function in P_4 is mapped to its derived function:

This space has dimension 4, and a basis is given below.

$$D: p \longmapsto p' \qquad (p \in P_4)$$

maps the space P_4 onto the space P_3 (which has dimension 3). In this case we could take as a basis for P_4 the set of functions:

$$\left.\begin{array}{ll} f_0: x \longmapsto 1 & (x \in R) \\ f_1: x \longmapsto x & (x \in R) \\ f_2: x \longmapsto x^2 & (x \in R) \\ f_3: x \longmapsto x^3 & (x \in R) \end{array}\right\} \quad \text{Basis for } P_4$$

This set of functions maps to the set

$$\begin{array}{ll} f_0': x \longmapsto 0 & (x \in R) \end{array}$$

$$\left.\begin{array}{ll} f_1': x \longmapsto 1 & (x \in R) \\ f_2': x \longmapsto 2x & (x \in R) \\ f_3': x \longmapsto 3x^2 & (x \in R) \end{array}\right\} \quad \text{Basis for } P_3$$

Like the previous example, this mapping is not one–one.

For example, all the functions of the form:

$$f:x \longmapsto x^2 + a \qquad (x \in R),$$

where $a \in R$, map to the function f_2'.

Note that f_0' is the zero vector in P_3. It cannot belong to any basis for P_3, since a basis must be a linearly independent set of three vectors. For consider the set $\{f_0', g, h\}$, where $g, h \in P_3$. Since

$$\alpha f_0' + 0g + 0h = f_0',$$

where α is any non-zero real number, we see that any set of (three) elements containing f_0' is linearly dependent.

Exercise

We seem to be putting a lot of faith in choosing a convenient basis. Is the choice of basis important? We shall resolve this difficulty later, but one point can be considered here.

In Example 3, instead of f_0 and f_1, we could choose g_0 and g_1, where

$$g_0:x \longmapsto 1 + x \qquad (x \in R)$$

$$g_1:x \longmapsto 1 - x \qquad (x \in R).$$

Then $\{g_0, g_1, f_2, f_3\}$ is another basis of P_4 and none of the basis vectors maps to the zero vector under D. Is $\{g_0', g_1', f_3'\}$ a basis for P_3?

Solution

$$g_0':x \longmapsto 1 \qquad (x \in R)$$

$$g_1':x \longmapsto -1 \qquad (x \in R).$$

Although none of the basis vectors is mapped to the zero vector, $\{g_0', g_1', f_3'\}$ is linearly dependent, since

$$1g_0' + 1g_1' + 0f_3' = f_0'.$$

g_0' and g_1' cannot both belong to the same basis because one is a scalar multiple of the other.

0.4.2 Linear Transformations

There are many interesting and useful results concerning mappings of vector spaces, but the most fruitful field of study consists of those mappings which are homomorphisms or isomorphisms. When we are considering a mathematical structure, it is often illuminating to study the functions which preserve that structure. A function which preserves structure is often called a *morphism*.

Let us first take a brief look at the additive structure of a vector space V, that is, the axioms *A1–A5*. We observe that this structure is that of a (commutative) group. It is not the purpose of this course to study groups; but we shall be well served by noting the underlying principle of studying a structure using a morphism. In this case, let us focus attention on a fixed scalar $\lambda \in R$, and consider the function

M101 Block VI *Unit 2*.

$$\mathbf{v} \longmapsto \lambda\mathbf{v} \qquad \text{for } \mathbf{v} \in V.$$

Axiom *B4* guarantees that the image of $\mathbf{v}_1 + \mathbf{v}_2$ under this function is given by

$$\mathbf{v}_1 + \mathbf{v}_2 \longmapsto \lambda\mathbf{v}_1 + \lambda\mathbf{v}_2,$$

since the right-hand side is equal to $\lambda(\mathbf{v}_1 + \mathbf{v}_2)$. In other words, this function is a *homomorphism* of the group structure of V, since the operation of addition is preserved.

Homomorphism: M101 Block VI *Unit 4*.

The full structure of the vector space V includes the operation of scalar multiplication; we shall, therefore, be concerned with functions on V which preserve *both* operations. Suppose a function T maps a vector space V, with an addition operation $+_V$, to a vector space U, with an addition operation $+_U$. The additive structure will be preserved if, for any vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V,

$$T(\mathbf{v}_1 +_V \mathbf{v}_2) = T(\mathbf{v}_1) +_U T(\mathbf{v}_2).$$

Since we have been abusing the symbol + through this unit (we have defined all sorts of methods of addition and called them all +), we shall continue to do so, and we drop the suffices U and V from the addition symbols.

We then have

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) \tag{1}$$

as the condition that T should be a morphism for the addition operations. For the other operation we require that

$$T(\alpha\mathbf{v}) = \alpha T(\mathbf{v}), \tag{2}$$

for any real number α and any vector $\mathbf{v} \in V$.

Equations (1) and (2) are the conditions that T should be a morphism from the vector space V to the vector space U.

The two equations can be combined to give the following equation:

$$T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) = \alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2), \tag{3}$$

for any real numbers α_1, α_2, and any vectors $\mathbf{v}_1$ and $\mathbf{v}_2 \in V$.

A mapping of a vector space to a vector space is often called a transformation, and when the mapping is a morphism it is called a linear transformation. This is another example of calling a particular type of mapping by a special name.

The significance of a linear transformation is that, given a basis for the domain, it is sufficient to know the images of the basis vectors—the images of all other vectors must follow. For example, suppose V is a vector space of dimension 3, with a basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$; then every vector $\mathbf{v} \in V$ can be expressed as

$$\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

for suitable scalars α, β, $\gamma \in R$.

Now if T is a linear transformation, then

$$T(\mathbf{v}) = \alpha T(\mathbf{a}) + \beta T(\mathbf{b}) + \gamma T(\mathbf{c}),$$

so if we know the three image vectors $T(\mathbf{a})$, $T(\mathbf{b})$ and $T(\mathbf{c})$, then we can deduce the image $T(\mathbf{v})$ for all $\mathbf{v} \in V$.

Exercises

Take the operations in the various vector spaces to be the usual ones.

1. Which of the following mappings are linear transformations?

   (i) The mapping of R^2 to R^2 such that

   $$T:(x_1, x_2) \longmapsto (x_2, x_1)$$

   (ii) The mapping of R^2 to R^2 such that

   $$T:(x_1, x_2) \longmapsto (x_1^2, x_2^2)$$

   (iii) The mapping of the set of all polynomial functions of degree n or less to itself such that

   $$T: p \longmapsto \text{the derived function of } p.$$

   (iv) The mapping of the set of all real functions which are twice-differentiable at all points in R to the set of all functions with domain R, such that

   $$T: f \longmapsto 2f'' + f' + 3f.$$

2. Let L be a linear transformation from a vector space V to a vector space U. Complete the gaps in the proof of the following theorem.

   THEOREM

   If the zero element of V is $\mathbf{v}_0$ (i.e. $\mathbf{v}_0$ is the element for which $\mathbf{v} + \mathbf{v}_0 = \mathbf{v}$ for any $\mathbf{v} \in V$), and if $\mathbf{u}_0$ is the zero element of U, then $L(\mathbf{v}_0) = \mathbf{u}_0$.

   PROOF

   Since $\mathbf{v} + \mathbf{v}_0 = \mathbf{v}$

   $$L(\mathbf{v} + \mathbf{v}_0) = L\left(\boxed{}\right) \qquad (a)$$

   But L is a linear transformation, so

   $$L(\mathbf{v} + \mathbf{v}_0) = L\left(\boxed{}\right) + L\left(\boxed{}\right) \qquad (b)$$

   From (a) and (b), $L(\mathbf{v}) + L(\mathbf{v}_0) = L(\mathbf{v})$, so, subtracting $L(\mathbf{v})$ from both sides, we see that

   $$L(\mathbf{v}_0) \text{ is the } \boxed{} \text{ vector of } U. \qquad (c)$$

   Confirm this result for each of the mappings which are linear transformations in Exercise 1.

Exercise 2 shows that under a linear transformation the zero element in the domain vector space is mapped to the zero element in the codomain vector space.

Solutions

1. (i)

$$\begin{aligned} T\big((x_1, x_2) + (y_1, y_2)\big) &= T(x_1 + y_1, x_2 + y_2) \\ &= (x_2 + y_2, x_1 + y_1) \\ &= (x_2, x_1) + (y_2, y_1) \\ &= T(x_1, x_2) + T(y_1, y_2). \end{aligned}$$

and

$$\begin{aligned} T\big(\alpha(x_1, x_2)\big) &= T(\alpha x_1, \alpha x_2) \\ &= (\alpha x_2, \alpha x_1) \\ &= \alpha(x_2, x_1) \\ &= \alpha T(x_1, x_2) \end{aligned}$$

so T is a morphism.

(ii) $$\alpha T(x_1, x_2) = \alpha(x_1^2, x_2^2) = (\alpha x_1^2, \alpha x_2^2),$$

and

$$T(\alpha(x_1, x_2)) = T(\alpha x_1, \alpha x_2) = (\alpha^2 x_1^2, \alpha^2 x_2^2).$$

Equation (2) is not satisfied, so T is not a morphism.

(iii) T is a morphism.

(iv) T is a morphism.

These results follow directly from the properties of differentiation.

2. (a) $\mathbf{v}$

(b) $\mathbf{v}, \mathbf{v}_0$

(c) zero

For the morphisms of Exercise 1, we have:

(i) $(0,0) \longmapsto (0,0)$

(iii) $(x \longmapsto 0) \longmapsto (x \longmapsto 0)$

(iv) $(x \longmapsto 0) \longmapsto (x \longmapsto 0)$

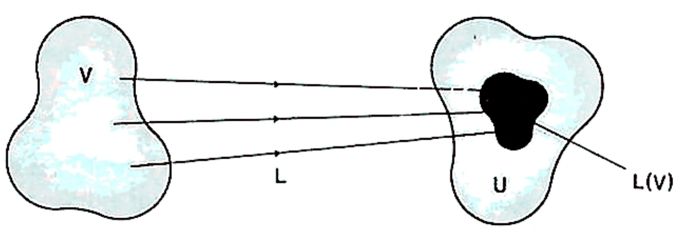
You may have noticed that, for mappings which are not one–one, the zero vector is not the only vector which maps to the zero vector. For example, in (iii) we also have $(x \longmapsto k)$ mapping to $(x \longmapsto 0)$, where k is any real number.

The following theorem is fundamental to the study of linear transformations. It provides the general framework for the three examples discussed in the preceding sub-section.

Theorem

If L is a linear transformation from a vector space V to a vector space U, then $L(V)$ is a subset of U which is itself a vector space.

Here $L(V)$ denotes the set of images $L(\mathbf{v})$ for all $\mathbf{v} \in V$; it may be U or a proper subset of U.



Method of Proof

We have to prove that the set $L(V)$, with the operations of the vector space U, satisfies the vector space axioms listed in Section 0.3. To save you the trouble of referring back, we list the axioms for a vector space V: $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are any elements of V and α and β are any real numbers:

A1 $\mathbf{v}_1 + \mathbf{v}_2 \in V$ and is unique.

A2 $\mathbf{v}_1 + (\mathbf{v}_1 + \mathbf{v}_3) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3$

A3 There is an element $\mathbf{v}_0$ in V, such that

$$\mathbf{v} + \mathbf{v}_0 = \mathbf{v}.$$

A4 Given $\mathbf{v} \in V$, there is an element $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{v}_0$.

A5 $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$.

B1 $\alpha\mathbf{v} \in V$

B2 $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$

B3 $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$

B4 $\alpha(\mathbf{v}_1 + \mathbf{v}_2) = (\alpha\mathbf{v}_1) + (\alpha\mathbf{v}_2)$

B5 $1 \times \mathbf{v} = \mathbf{v}$.

Axioms *A2*, *A5*, *B2*, *B3*, *B4* and *B5* are statements about *all* elements of a vector space, and since U is a vector space we do not have to check these axioms for $L(V)$. On the other hand, axiom *A3* is a statement that a particular kind of element (the ZERO element) belongs to a vector space. Clearly, the zero element of U will not belong to every subset of U, so we

have to prove that it belongs to the particular subset $L(V)$. Axioms $A1$ and $B1$ concern CLOSURE. If $L(V)$ is to be a vector space, then any combination of elements in $L(V)$ must give resulting elements still in $L(V)$. This again is not necessarily true for any subset of U, so we *must* check it for $L(V)$. Once we have proved $B1$, we shall not need to prove $A4$, because we always have $(-1)\mathbf{v} = -\mathbf{v}$.

Proof

We have to prove that axioms $A1$, $A3$ and $B1$ hold for $L(V)$. We have three pieces of information:

(i) V is a vector space;
(ii) U is a vector space;
(iii) L is a linear transformation.

We have used (ii) to dispose of axioms $A2$, $A5$, $B2$, $B3$ and $B5$ but we have not yet used (i) and (iii).

We have proved that axiom $A3$ holds for $L(V)$: in Exercise 2 we proved that under a linear transformation the zero vector $\mathbf{v}_0$ in the domain V maps to the zero vector $\mathbf{u}_0$ in the codomain U. So the zero vector of U belongs to $L(V)$.

Let us have a look at axiom $A1$; we must show that $L(V)$ is closed under addition. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are any elements of $L(V)$, then there are elements $\mathbf{v}_1$ and $\mathbf{v}_2$ in V such that

$$\mathbf{u}_1 = L(\mathbf{v}_1)$$

$$\mathbf{u}_2 = L(\mathbf{v}_2).$$

Then

$$\begin{aligned}\mathbf{u}_1 + \mathbf{u}_2 &= L(\mathbf{v}_1) + L(\mathbf{v}_2)\\ &= L(\mathbf{v}_1 + \mathbf{v}_2) \qquad \text{(because } L \text{ is a linear transformation)}\\ &= L(\mathbf{v}_3),\end{aligned}$$

where $\mathbf{v}_3$ is an element of V (by axiom $A1$ for the vector space V): $L(\mathbf{v}_3)$ is an element of $L(V)$, so $\mathbf{u}_1 + \mathbf{u}_2$ belongs to $L(V)$, and $L(V)$ is closed.

The other closure axiom $B1$ is easily checked.

If

$$\mathbf{u} = L(\mathbf{v}),$$

then

$$\begin{aligned}\alpha\mathbf{u} &= \alpha L(\mathbf{v})\\ &= L(\alpha\mathbf{v}) \qquad \text{(because } L \text{ is a linear transformation)}\end{aligned}$$

Therefore $\alpha\mathbf{u} \in L(V)$.

This completes the proof.

Note that we have not proved that if $T: V \longmapsto U$ is *not* a morphism, then $T(V)$ is *not* a vector space. For a general mapping T, we do not know anything about $T(V)$ if T is not a morphism.

If a subset of a vector space U is itself a vector space, then we call it a vector subspace of U. So we have shown in the theorem that the image of a linear transformation is a vector subspace of the codomain.

Exercise

3. (i) The mapping from R^2 to R^2 defined by

$$L: (x_1, x_2) \longmapsto (-x_2, x_2)$$

is a linear transformation. Prove directly, by verifying the axioms, that $L(R^2)$ is a vector space.

(ii) The mapping from R^2 to R^2 defined by

$$T:(x_1,x_2) \longmapsto (x_1{}^2, x_2)$$

is not a linear transformation. Show that $T(R^2)$ is not a vector space by finding an axiom which is not satisfied.

Solution

3. (i) As in the proof of the theorem, we need to check axioms *A1*, *A3* and *B1* only.

A1 The elements of $L(R^2)$ are of the form

$$(-a, a), \quad a \in R$$

and

$$(-a,a)+(-b,b) = (-(a+b), a+b),$$

and so axiom *A1* is satisfied.

A3 Consider the element $(0,0)$

$$L((0,0)) = (0,0),$$

so that $(0,0) \in L(R^2)$ and axiom *A3* is satisfied.

B1 Any element of $L(R^2)$ is of the form $(-a,a)$ for some $a \in R$;

$$\alpha(-a,a) = (-\alpha a, \alpha a),$$

and so $\alpha(-a,a) \in L(R^2)$: axiom $B1$ is satisfied.

(ii) The only axioms which may not be satisfied are *A1*, *A3* and *B1*. Of these only *B1* is not satisfied.

$$\alpha(x_1^2, x_2) = (\alpha x_1^2, \alpha x_2)$$

and if α is negative, αx_1^2 is also negative, and so cannot be written as the square of a real number. We see therefore that for the given function $T: R^2 \longmapsto R^2$, the conclusion of the theorem is violated; we may therefore deduce that T is not a linear transformation.

The subset of R^2 which we encountered as $T(R^2)$ in the exercise is in fact the *half-plane*

$$\{(x_1,x_2): x_1 \geq 0\}.$$

It is a convenient example of a subset which is *not* a vector subspace.

So far in the text, we have considered mappings of one vector space to another and we have concentrated our attention on linear transformations, i.e., those mappings which are morphisms. A linear transformation has the property that the image set itself is a vector space.

An interesting feature of some of the morphisms we have met is that they map a vector space on to an image set which has a lower dimension. For example, we have had mappings of planes to lines, polynomials of degree n or less to polynomials of degree $n-1$ or less, and so on. Two questions arise. What has happened to the "lost" dimensions? Can we predict in advance when we are going to "lose" a dimension? We shall look at these questions in the next sub-section.

0.4.3 The Kernel

Let us have another look at the linear transformation

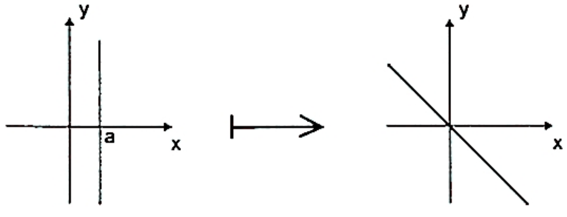
$$L:(x,y) \longmapsto (-y,y) \qquad \text{for } (x,y) \in R^2$$

which maps the plane R^2 to a line. In sub-section 0.4.1, we looked at a particular basis, and saw that one of the basis vectors mapped to the zero

element $(0,0)$ in the codomain. We then investigated the mapping by looking to see what happened to particular subsets of the plane. We saw that any line parallel to the x-axis mapped to a single point.

This raises two questions. Firstly, is it significant that we lose one basis vector and we lose one dimension? Secondly, it is all very well in this simple case to pick out a few significant subsets that tell us such a lot. We picked them out because we knew their properties. Consider now the images of the lines parallel to the y-axis in the domain. Any such line maps to the entire image set, for suppose we take the line for which $x = a$, then

$$L:(a,y) \longmapsto (-y,y) \qquad (y \in R).$$



By considering certain subsets of the domain, we find that we can obtain information about L. Is there any particular subset which we can most profitably consider? That is, can we describe a subset in the domain which will give us information about L in a form which we can interpret easily? If so, can we extract any general feature which will help us with other examples?

The clue is in our observation about the loss of a basis vector. The vector $(1,0)$ maps to $(0,0)$, but of course it is not only this vector which "shrinks" to zero—so does every multiple of $(1,0)$. So a whole set maps to $(0,0)$. Why consider this particular set? Try the next exercise.

Because the mapping is a linear transformation,

$$L(\alpha(1,0)) = \alpha L(1,0) = (0,0).$$

Exercise

1. The vectors $(0,1)$ and $(2,2)$ form a basis for R^2. Calculate $L(0,1)$ and $L(2,2)$, where L is the mapping we have been discussing:

$$L:(x,y) \longmapsto (-y,y) \qquad ((x,y) \in R^2).$$

What happens to the linear independence of $(0,1)$ and $(2,2)$?

Solution

1. $$L(0,1) = (-1,1) \neq (0,0)$$

and

$$L(2,2) = (-2,2) \neq (0,0)$$

but

$$-2L(0,1) + L(2,2) = (0,0).$$

Although neither vector maps to zero, the pair of linearly independent vectors maps to a pair of dependent vectors. So although the original vectors form a basis for R^2, their images do not.

Exercise 1 shows us that, in our original basis, the choice of a vector which mapped to $(0,0)$ was purely fortuitous. It may so happen, as in this exercise, that none of the basis vectors maps to $(0,0)$, even though we "lose" a dimension. But the whole set $\{(\alpha,0):\alpha \in R\}$ maps to $(0,0)$ whether or not one of its elements is in the basis.

It seems then that the set which maps to $(0,0)$ tells us something about the "lost" dimension. In this case the set which maps to $(0,0)$ has dimension 1 (every element of the set can be obtained as a scalar multiple of the vector

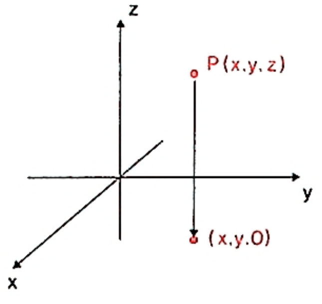
$\mathbf{i} = (1, 0)$), and we lose just one dimension. Let us have a look at two more examples, one where we again lose one dimension and one where we lose more than one dimension. We shall again take examples which can be modelled geometrically because geometrical situations are easy to visualize.

Example 1

The mapping

$$L: (x, y, z) \longmapsto (x, y, 0)$$

is a linear transformation from R^3 to R^3. The image of any point P is the point at the foot of the perpendicular from P to the plane with equation $z = 0$. Thus the domain maps to a plane.



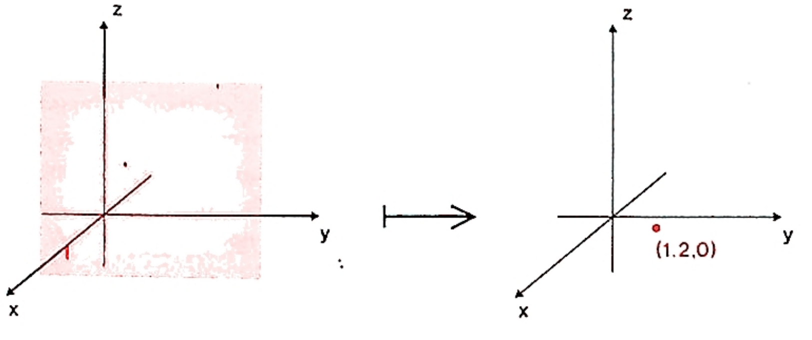
L maps a 3-dimensional space to a 2-dimensional space: we lose one dimension. The set which maps to the zero element $(0, 0, 0)$ in the codomain is the set $\{(0, 0, z): z \in R\}$, that is, the z-axis. This set is itself a vector space and its dimension is one, the same number as the number of "lost" dimensions.

Example 2

The mapping

$$L: (x, y, z) \longmapsto (x, 2x, 0)$$

is a linear transformation from R^3 to R^3. The image of the point (x, y, z) depends only on its x-coordinate. Thus $(1, 2, 3)$, $(1, 6, 7)$, $(1, 6, 99)$ all map to the point $(1, 2, 0)$. Every point in the plane with equation $x = 1$ maps to this point.

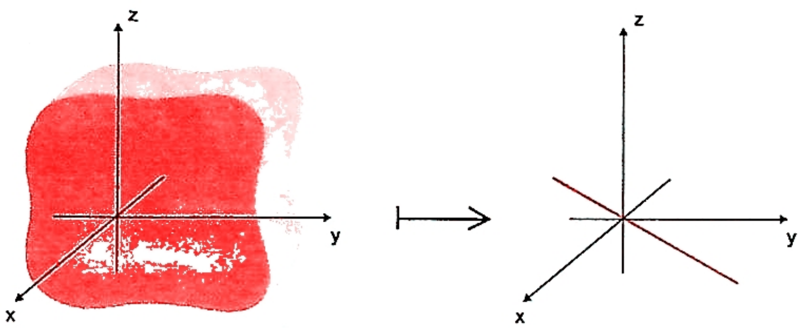


Similarly, every point on the plane with equation $x = 2$ maps to the point $(2, 4, 0)$ and so on. Every plane perpendicular to the x-axis maps to a point on the line defined by the equations:

$$2x - y = 0$$
$$z = 0,$$

In three-dimensional Cartesian space, two linear equations are required to determine a line.

and the entire three-dimensional space maps to this complete line.



Thus

$$L(R^3) = \{(x, y, z) : 2x - y = 0, z = 0\}.$$

The three-dimensional domain maps to a space of dimension 1. We seem to have lost two dimensions.

Which set maps to the zero element? In this case the zero element is $(0, 0, 0)$, and the set which maps to $(0, 0, 0)$ is the set $\{(x, y, z) : x = 0\}$, i.e. the yz-plane: this is itself a vector space and has dimension two. Notice that in this simple case we can use the "basis argument" again—if we can find an appropriate basis. Taking the set of vectors $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ as a basis, we see that both $(0, 1, 0)$ and $(0, 0, 1)$ map to $(0, 0, 0)$, and so we "lose" two vectors from the basis. In fact we "lose" any vector which can be expressed as a linear combination of these two vectors (i.e. the points in the plane with equation $x = 0$), because

$$\begin{aligned} L(\alpha(0, 1, 0) + \beta(0, 0, 1)) &= \alpha L(0, 1, 0) + \beta L(0, 0, 1) \\ &= (0, 0, 0), \end{aligned}$$

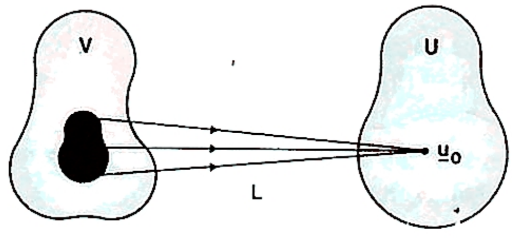
since L is a linear transformation.

We have seen that the subset of the domain which maps to the zero vector in the codomain plays an important part, so we now give it a name.

If L is a linear transformation from a vector space V to a vector space U, and if $\mathbf{0}_U$ is the zero element in U, then the set

$$\{\mathbf{v} : \mathbf{v} \in V, L(\mathbf{v}) = \mathbf{0}_U\}$$

is called the kernel of L. (Another name which is in common use for this set is the *null space*.) We shall denote the kernel by the letter K.



There is one important point to notice here. In sub-section 0.3.3 we defined a basis of a vector space V to be a *linearly independent* set of vectors in V which *spans* V. We defined the dimension of V to be the number of elements in a basis. Now the kernel of a linear transformation might only contain the zero element of V; i.e. we may have $K = \{\mathbf{0}\}$.

We write $\mathbf{0}$ instead of $\mathbf{0}_V$ here, because you may like to refer to our discussion of the vector space $\{\mathbf{0}\}$, where we mentioned that

$$\alpha\mathbf{0} = \mathbf{0},$$

where α is any real number. This means that $\{\mathbf{0}\}$ is a linearly dependent set, that is $\{\mathbf{0}\}$ does not possess a basis. We adopt the following definition.

The dimension of the zero vector space, $\{\mathbf{0}\}$, is zero.

The kernel has some quite remarkable properties. We have already hinted at two of them which are printed in red below.

(1) The kernel itself is a vector space.

We have shown that $L(V)$ is itself a vector space, and in Examples 1 and 2 we have seen that

(2) (dimension of $L(V)$) = (dimension of V) − (dimension of kernel).

Exercises

2. Find the kernel of each of the following linear transformations.

   (i) $T:(x_1, x_2, x_3) \longmapsto (x_1, x_2, 0)$

   where T maps R^3 to R^3.

   (ii) $T:(x_1, x_2) \longmapsto (x_1 + x_2, x_1 - 2x_2)$

   where T maps R^2 to R^2.

   In each case find the dimension of the kernel and verify statement 2 in the text.

3. Let L be a linear transformation from a vector space V to a vector space U. Show that the kernel of L is a vector subspace of V.

   HINT: How many of the axioms of a vector space need proving for the kernel? (See sub-section 0.4.2.)

Solutions

2. (i) $\{(0, 0, x_3): x_3 \in R\}$. Any element in this set is a scalar multiple of $(0, 0, 1)$.

   (ii) $\{(x_1, x_2): x_1 + x_2 = 0 \text{ and } x_1 - 2x_2 = 0\}$.

   The pair of simultaneous equations

$$x_1 + x_2 = 0$$
$$x_1 - 2x_2 = 0$$

   has the single solution $x_1 = 0$, $x_2 = 0$. Thus the kernel is the set $\{(0, 0)\}$.

   The dimensions of the kernels are as follows:

   (i) 1. The dimension of the domain is 3, the dimension of its image set is 2, and $3 - 2 = 1$.

   (ii) 0. The dimension of both the domain and its image is 2. Note that, by defining the dimension of $\{\mathbf{0}\}$ to be zero, we have ensured that statement 2 is satisfied when $K = \{\mathbf{0}\}$.

3. We need only prove that the elements of the kernel satisfy axioms *A1*, *A3* and *B1*. As usual, we denote the kernel by K.

   A1 If $\mathbf{k}_1$ and $\mathbf{k}_2$ belong to K, we want to prove that $\mathbf{k}_1 + \mathbf{k}_2 \in K$. We recognize elements of K by the fact that under L they map to $\mathbf{0}_U$, the zero element in U.

$$\begin{aligned} L(\mathbf{k}_1 + \mathbf{k}_2) &= L(\mathbf{k}_1) + L(\mathbf{k}_2) && (L \text{ is a linear transformation}) \\ &= \mathbf{0}_U + \mathbf{0}_U && (k_1, k_2 \in K) \\ &= \mathbf{0}_U && (\text{axiom } A3 \text{ for } U) \end{aligned}$$

   Therefore, $\mathbf{k}_1 + \mathbf{k}_2 \in K$.

   A3 We have already shown that $L(\mathbf{0}) = \mathbf{0}_U$.

   Therefore, $\mathbf{0} \in K$.

B1 Let $\mathbf{k} \in K$, then

$$\begin{aligned} L(\alpha\mathbf{k}) &= \alpha L(\mathbf{k}) && (L \text{ is a linear transformation}) \\ &= \alpha\mathbf{0}_U && (k \in K) \\ &= \mathbf{0}_U && (\text{property of the zero vector}) \end{aligned}$$

The verification of axioms *A1* and *B1* can be amalgamated by using the definition of a linear transformation in the form given in Equation (3) of sub-section 0.4.2.

0.4.4 Properties of the Kernel

It is remarkable how much we can tell about a linear transformation just by considering its kernel.

Let L be a linear transformation from a vector space V to a vector space U, and let K be its kernel. If $\mathbf{k} \in K$ and $\mathbf{v} \in V$, then

$$\begin{aligned} L(\mathbf{v} + \mathbf{k}) &= L(\mathbf{v}) + L(\mathbf{k}) && (L \text{ is a morphism}) \\ &= L(\mathbf{v}) + \mathbf{0}_U && (\text{definition of } K) \\ &= L(\mathbf{v}) && (\text{axiom } A3 \text{ for } U), \end{aligned}$$

where $\mathbf{0}_U$ is the zero element in U. So $\mathbf{v}$ and $\mathbf{v} + \mathbf{k}$, where $\mathbf{k}$ is any element of the kernel, have the same image.

Suppose now that we want to find all the elements in V which map to a given element $\mathbf{u} \in U$, and that we know one such element $\mathbf{v}$, i.e.

$$L(\mathbf{v}) = \mathbf{u}.$$

Then we know immediately that $\mathbf{v} + \mathbf{k}$ for all $\mathbf{k} \in K$ are such elements, and the remarkable thing is that they are in fact all the elements which map to $\mathbf{u}$. We can prove this as follows. Suppose $\mathbf{v}_1 \in V$ maps to $\mathbf{u}$, i.e. $L(\mathbf{v}_1) = \mathbf{u}$. Then consider $\mathbf{v}_1 - \mathbf{v}$. We have

$$\begin{aligned} L(\mathbf{v}_1 - \mathbf{v}) &= L(\mathbf{v}_1 + (-\mathbf{v})) && (\text{definition of subtraction}) \\ &= L(\mathbf{v}_1) + L(-\mathbf{v}) && (L \text{ is linear}) \\ &= L(\mathbf{v}_1) - L(\mathbf{v}) && (L \text{ is linear}) \\ &= \mathbf{u} - \mathbf{u} && (\text{hypothesis}) \\ &= \mathbf{0}_U && (\text{axiom } A4 \text{ for } U). \end{aligned}$$

But the kernel K contains all those elements which map to $\mathbf{0}_U$, so

$$\mathbf{v}_1 - \mathbf{v} = \mathbf{k}_1$$

for some $\mathbf{k}_1 \in K$. By axiom *A1* for V, $\mathbf{k}_1$ is unique. Adding $\mathbf{v}$ to both sides, we get

$$\mathbf{v}_1 = \mathbf{v} + \mathbf{k}_1,$$

and so $\mathbf{v}_1$ is of the form $\mathbf{v}$ + some element of the kernel. This result has important consequences: for instance, if the kernel contains n linearly independent elements $\mathbf{k}_1, \ldots, \mathbf{k}_n$ then we know that if $\mathbf{v} \in V$ maps to any given element of $L(V)$, then so also do all vectors of the form

$$\mathbf{v} + \lambda_1\mathbf{k}_1 + \lambda_2\mathbf{k}_2 + \cdots + \lambda_n\mathbf{k}_n.$$

Furthermore, if the kernel contains just one element (which will necessarily be the zero element in V), then we know immediately that L is one-one, i.e. an isomorphism. The following example applies this discussion.

Example 1

Apply the above ideas to finding the solution of the equations

$$2x + 3y - z = 1$$
$$x + y - z = 2$$

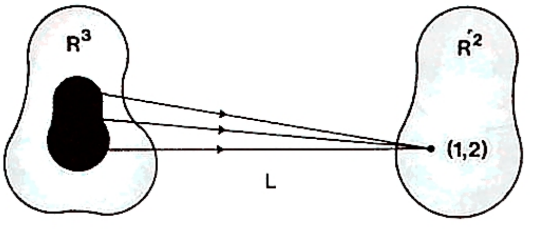
in terms of vector spaces.

Solution of Example 1

One way of expressing the problem is to say that we want to find the set of triples (x, y, z) which satisfy these equations. If L is the mapping from R^3 to R^2 defined by

$$L:(x, y, z) \longmapsto (2x + 3y - z, x + y - z),$$

then we want to find the set which maps to $(1, 2)$.



We have seen that we can describe the set which maps to any particular element when we know the kernel and one element of the set. In the context of this example, this means that, if we can find one solution to the equations, we shall be able to find *all* the solutions, simply by adding to that solution each element of the kernel. So we have to find *one solution* and we have to find the *kernel*.

One Solution

If we give x or y or z a particular value, then the equations will be reduced to two equations in two unknowns, which we can solve easily.

For example, if we put $z = 0$, then we obtain the two equations

$$2x + 3y = 1$$
$$x + y = 2,$$

which we can solve to give

$$x = 5, \qquad y = -3.$$

So one solution of the original equations

$$2x + 3y - z = 1$$
$$x + y - z = 2$$

is

$$x = 5, \quad y = -3, \quad z = 0.$$

We call this a *particular solution* of the equations.

The Kernel

The kernel, K, is the set of triples (x, y, z) which map to $(0, 0)$, i.e. which satisfy the equations

$$2x + 3y - z = 0$$
$$x + y - z = 0$$

We call these the *associated homogeneous equations.*

Just as before, we can solve these equations by giving x or y or z a particular value and then trying to solve the resulting two equations in two unknowns; but this time it is not much help, because we simply get the *one* solution, and

we want *all* solutions. But if, for example, we give z a general value and put $z = k$, then these equations become

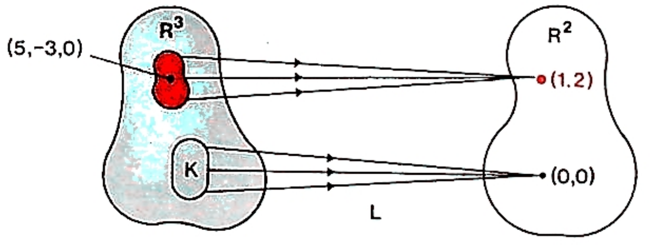
$$2x + 3y = k$$
$$x + y = k,$$

which we can solve to give

$$x = 2k, \qquad y = -k.$$

So one element of the kernel is $(2k, -k, k)$, and by varying k we get all the elements. So K is the set

$$\{(2k, -k, k): k \in R\}.$$



We want to find the set shaded red in the above diagram.

We know one element—how do we find the others?

Any solution of the original equations

$$2x + 3y - z = 1$$
$$x + y - z = 2$$

is obtained by adding an element of the kernel to $(5, -3, 0)$.

So the complete solution set is

$$\{(5 + 2k, -3 - k, k): k \in R\},$$

and the theory we have developed assures us that this set contains *all* the possible solutions to the original equations.

We say that

$$x = 5 + 2k, \quad y = -3 - k, \quad z = k$$

is the *general solution*.

(Check that these *are* solutions by substituting into the original equations.)

Notice that we can solve related problems like

$$2x + 3y - z = 7$$
$$x + y - z = 99,$$

where the right-hand sides of the equations are changed, very quickly—all we need to do is to find one particular solution of *these* equations and then add on each of the elements of the (same) kernel. We shall discuss problems of this type in considerable detail in *Unit 3, Hermite Normal Form.*

Exercises

1. By putting $x = 0$, find a particular solution of the equations

$$2x + 3y - z = 2$$
$$x + y + z = 1$$

Find the solution set of the equations.

2. We can map the vector space P_n, of all polynomial functions of degree *less than* n, to itself by the differentiation operator:

$$D: p \longmapsto p' \qquad (p \in P_n).$$

We have already seen that this mapping is linear. What is the kernel?

What significance does this have in integration?

Solutions

1. Putting $x = 0$ in both equations, gives

$$3y - z = 2$$
$$y + z = 1,$$

which have the solution

$$y = \tfrac{3}{4}, \quad z = \tfrac{1}{4}$$

Thus one solution is

$$x = 0, \quad y = \tfrac{3}{4}, \quad z = \tfrac{1}{4}.$$

To find the kernel, we have to solve the equations

$$2x + 3y - z = 0$$
$$x + y + z = 0$$

If we put $x = k$, we get

$$3y - z = -2k$$
$$y + z = -k,$$

which have the solution

$$y = -\tfrac{3}{4}k, \quad z = -\tfrac{1}{4}k;$$

so the kernel is the set

$$\{(k, -\tfrac{3}{4}k, -\tfrac{1}{4}k) : k \in R\}.$$

The complete solution is therefore given by

$$\{(k, \tfrac{3}{4} - \tfrac{3}{4}k, \tfrac{1}{4} - \tfrac{1}{4}k) : k \in R\}.$$

The equations $2x + 3y - z = 2$ and $x + y + z = 1$ are equations of planes. The set of points

$$\{(k, \tfrac{3}{4} - \tfrac{3}{4}k, \tfrac{1}{4} - \tfrac{1}{4}k) : k \in R\}$$

corresponds to the line formed by the intersection of these two planes.

2. The kernel is the set of all polynomial functions which map to the zero function

$$x \longmapsto 0 \qquad \text{for } x \in R;$$

that is, the set of all constant functions,

$$\{f : x \longmapsto k \qquad (x \in R), \text{ where } k \in R\}.$$

The problem of integration is that of solving equations of the form

$$D(p) = g,$$

where g is given.

Since the kernel contains an infinite number of elements, the integration problem has an infinite number of solutions. If p is one solution, then the set of all solutions is

$$\{p + f : f : x \longmapsto k \qquad (x \in R), \text{ where } k \in R\}.$$

Problems such as the following often arise in mathematics. We are given a mapping, M say, from a set A to a set B, and are required to find all the elements x of A such that

$$M : x \longmapsto b,$$

where b is a given element of B. That is to say, we have to solve the equation $M(x) = b$. Sometimes there is no solution (if b does not belong to $M(A)$); sometimes there is a unique solution; sometimes there are many solutions. In a very wide class of problems, A and B are vector spaces and M is a linear transformation. We have just seen that in these cases we can (in principle) adopt a standard procedure. We first find the kernel, K, that is, the solution set of

$$M(\mathbf{x}) = \mathbf{0}.$$

This is usually a much easier problem than the original one.

We then find any one particular solution of

$$M(\mathbf{x}) = \mathbf{b},$$

and then combine this with each element of K, to get the complete solution set.

We shall apply these techniques later in the course.

0.4.5 Summary of Section 0.4

In this section we defined the terms

morphism	(page 39)
linear transformation	(page 40)
subspace	(page 43)
kernel	(page 47)
associated homogeneous equations	(page 50)

We introduced the notation

$L(V)$	(page 42)
K	(page 47)
P_n	(page 51)

0.5 SUMMARY OF THE UNIT

We have seen how the concept of a vector space may be regarded as having its origins in geometry. Using the concept of a geometric vector, we were able to construct an algebraic structure by introducing "addition" and "multiplication by a scalar" on the set of all geometric vectors in two (or three) dimensions. We then saw that we could construct a very similar algebraic structure on the set of ordered pairs (or triples) of real numbers. There are in fact many different mathematical systems which have the same structure, and we chose to extract the important properties, then to study the abstract structure which possesses these properties. Such a structure we called a vector space. The concept of a vector space is of fundamental importance; it is so important that the whole course is built around it.

We then discussed what happens when vector spaces are mapped to vector spaces. This led us to the concept of a linear transformation.

In the next two units of the course, you will meet again much of what we have done on vector spaces and linear transformations. The subject will be put on firm foundations and many of the unproved statements in this unit will be verified.

LINEAR MATHEMATICS

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